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On the conformal field theory of the two-dimensional Ising model

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Abstract. An explicit construction of a free and masseless Majorana quantum field theory, which exists on a conformal superworld $\mathcal{M}$ is presented. Emphasis is placed on the investigation of the action of an infinite-dimensional group $\mathcal{G}$ of spacetime symmetries on $\mathcal{M}$. Starting with a one-particle theory, this action induces a strongly continuous representation of $\mathrm{Diff}_+^\sim(S^1)$ on the single-particle Hilbert space. After second quantization the group of implementers turns out to be a non-trivial central extension of $\mathrm{Diff}_+^\sim(S^1)$ by $\mathrm{U}(1)$, and a Schwinger term occurs, which gives rise to the anomalous transformation law of the energy–momentum tensor of the theory. This transformation law is studied, and an interesting connection to geometry is established.

1. Introduction and summary of results

At its critical point the two-dimensional Ising model is described by a free masseless Majorana field theory which possesses a symmetric and—due to scale invariance of the underlying statistical model at T_c—traceless canonical energy–momentum tensor $\Theta^{\mu\nu}$. For an early and physically clearcut approach which uses a doubled version of the Ising model based on a complex Dirac field, and which allows a simple description of the scale invariant limit see Schroer and Truong (1978). The Schwinger functions of this free field theory obey the conformal Ward identities which implies invariance under the Euclidean conformal group in two dimensions (Saint Aubin 1987, Miwa 1984, O'Carrol and Schor 1982). For this reason it is natural to regard the Majorana field as an object existing on the so-called conformal superworld $\mathcal{M}$, which is nothing but the universal covering space of compactified Minkowski space M^c (Mack 1987). Taking this as a motivation we present in this paper an explicit and mathematically rigorous construction of such a free masseless Majorana field theory on $\mathcal{M}$ which—when second quantized—exactly reproduces the Majorana field theory of the two-dimensional Ising model. Although emphasis is laid upon an algebraic treatment, it turns out that there is a very interesting interplay between geometrical and functional analytical methods in this approach.

The most intriguing fact about $(1+1)$-dimensional conformal kinematics is the existence of an infinite-dimensional Fréchet Lie group $\mathcal{G}$ which acts as a group of spacetime symmetries on $\mathcal{M}$ and which contains the Minkowskian conformal group as a subgroup (Schomerus and Mack 1990, Schroer 1988). We shall take this classical group action, which is briefly sketched in section 2, as a starting point. In section 3

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the classical field theory of a two-component free and masseless Majorana field is introduced on $\mathcal{M}$ within a Lagrangian formulation. Classical field configurations correspond to smooth sections of a vector bundle $\mathcal{E}$ over $\mathcal{M}$. Due to factorization of the classical theory in light cone coordinates in a theory of right and left movers, the chiral components of the Majorana field, which are solutions of the zero mass Dirac equation obeying antiperiodic boundary conditions at 'infinity', can be treated independently. They turn out to be in one-to-one correspondence to smooth sections of the Möbius strip. In section 4 this one-to-one correspondence is used to extract a complex single-particle Hilbert space $\mathcal{H}$ together with an antilinear involution C which implements the reality condition of the Majorana field configurations on $\mathcal{H}$. Moreover a unitary representation of the twofold covering $\mathrm{Diff}^{\sim}_{+}(S^1)$ of the connected component $\mathrm{Diff}_{+}(S^1)$ of the diffeomorphism group of the circle is found which commutes with C. Strong continuity of this representation is proved, and a Lie algebra of self-adjoint differential operators is obtained, which is interpreted as the Lie algebra of observables of the single-particle theory. Each vector field ξ in $\mathrm{Vect}(S^1)$ is associated with an observable D_ξ, such that the mapping $\xi \mapsto D_\xi$ is a Lie algebra homomorphism. The differential operator which corresponds to the constant vector field is identified with the conformal Hamiltonian denoted by D_0, which has an entirely discrete spectrum $\mathbb{Z} + \frac{1}{2}$. Its spectral projection P on the positive energy subspace of D_0 satisfies $CP = (\mathbb{1} - P)C$, and with this single-particle theory at hand second quantization, which is performed in the spirit of Segel and Mackey (1963) in section 5, is straightforward. Strictly speaking associated with the pair $(\mathcal{H}, C)$ there is a uniquely defined self-dual C^* algebra $\mathfrak{A}(\mathcal{H}, C)$, on which the previously mentioned group acts via Bogoliubov automorphisms. This action is continuous provided the automorphism group carries the strong operator topology. The orthogonal projection P gives rise to an $\mathrm{SU}(1,1)$-invariant Fock state over $\mathfrak{A}(\mathcal{H}, C)$, which implies that this symmetry is unitarily implemented in the GNS representation. In section 6 the general implementation problem is discussed, and the necessary and sufficient Hilbert–Schmidt condition for implementation of Bogoliubov automorphisms on Fock space is proved for all automorphisms under consideration. The so obtained group of implementers is a central extension of $\mathrm{Diff}^{\sim}_{+}(S^1)$ by $\mathrm{U}(1)$, which acts reducibly on Fock space. Restriction of this group to the even-particle subspace $\mathcal{F}_{\mathrm{ev}}(P\mathcal{H})$ yields a central extension of $\mathrm{Diff}_{+}(S^1)$, acting irreducibly on $\mathcal{F}_{\mathrm{ev}}(P\mathcal{H})$. The smallest C^* algebra which is generated by these unitaries on $\mathfrak{B}(\mathcal{F}_{\mathrm{ev}}(P\mathcal{H}))$ is then identified with the algebra $\mathcal{O}$ of chiral observables. It is interesting to see that $\mathcal{O}$ can be obtained by an abstract construction similar to that of the Weyl algebra. Recently Schroer has pointed out that the physical meaning of those diffeomorphisms which do not come from a Möbius transformation is connected with the Tomita–Takesaki theory for non-connected intervals. For connected intervals one obtains the Möbius group.

Whereas implementation of single Bogoliubov automorphisms is dealt with in section 6, implementation of one-parameter groups requires additional work, and in section 7 it is proved that an arbitrary one-parameter group $\{e^{iD_\xi t} : t \in \mathbb{R}\}$ in $\mathrm{Diff}^{\sim}_{+}(S^1)$ admits a lift to a strongly continuous one-parameter group $\{e^{i\Theta(\xi)t} : t \in \mathbb{R}\}$ on Fock space. The Lie algebra of the self-adjoint generators $\{\Theta(\xi) : \xi \in \mathrm{Vect}(S^1)\}$ yields to a non-trivial central extension of the Lie algebra of vector fields on the circle, and its complexification contains the Virasoro algebra as a subalgebra. The value of $c = \frac{1}{2}$ of the conformal charge can be easily read from the explicit form of the Schwinger term. Moreover it turns out that for φ', φ in a dense domain in Fock space, $\xi \mapsto \langle\varphi', \Theta(\xi)\varphi\rangle$ is a continuous linear functional, which demonstrates that the energy–momentum tensor in the second quantized theory is indeed an operator-valued distribution. In section 8 the anomalous transformation law of the energy–momentum tensor is studied in detail. For this purpose the adjoint action of $\mathrm{Diff}_+(S^1)$ on the central extension of $\mathrm{Vect}(S^1)$ is computed, and a nonlinear third-order differential operator, Δ on $\widetilde{\mathrm{Diff}}_+(S^1)$, which is closely related to the Schwarzian derivative, is obtained. Finally the connection with a more geometrical point of view found in Segal (1981) is established.

2. Kinematics

2.1. General considerations

A detailed investigation of analytic continuation in the case of conformal symmetries has revealed the fact that whenever the Schwinger functions of an underlying Wightman field theory exhibit invariance under infinitesimal transformations of the Euclidean conformal group $\mathrm{SO}(d+1,1)$, its Wightman functions admit analytic continuation to a domain of holomorphy which has as a real boundary an infinite sheeted covering space $\mathcal{M} \simeq \mathbb{R} \times S^{d-1}$ of compactified Minkowski space M^c (Lüscher and Mack 1975, Mack 1987). Because of this $\mathcal{M}$ is called the conformal superworld. In addition there is an action of an infinite sheeted covering group of the conformal group $\mathrm{SO}(d,2)/\mathbb{Z}_2$ on $\mathcal{M}$, which respects the causal ordering. Special conformal transformations are well defined and Einstein causality is restored for the price that $\mathcal{M}$ contains an infinite number of identical copies of Minkowski spaces, which is described in picturesque language as an 'infinite staircase of heavens and hells'. Thus one is led to the conclusion that a d-dimensional conformal quantum field theory exists on $\mathcal{M}$ rather than on Minkowski space.

In this paper we start with the $(1+1)$-dimensional Minkowski space $M = \mathbb{R}^2$ with the standard metric $\mathrm{d}x^0 \otimes \mathrm{d}x^0 - \mathrm{d}x^1 \otimes \mathrm{d}x^1$. Then M has compactification M^c, which is covered by $\mathcal{M} = \mathbb{R} \times S^\infty$. It is appropriate to think of $\mathcal{M}$ as the submanifold

$$\mathcal{M} = \{(\sigma^0, \cos\sigma^1, \sin\sigma^1) |\ \sigma^0 \in \mathbb{R}, \sigma^1 \in (-\pi, \pi]\} \subset \mathbb{R}^3 \quad . \tag{2.1}$$

with coordinates (σ^0, σ^1). A global causal structure on $\mathcal{M}$ is induced by the Lorenz metric $g = \mathrm{d}\sigma^0 \otimes \mathrm{d}\sigma^0 - \mathrm{d}\sigma^1 \otimes \mathrm{d}\sigma^1$. It coincides with the causal structure on $\mathcal{M}$ inherited from the pre-image of M under the surjection $\mathcal{M} \longrightarrow M$ which is defined by $(\sigma^0, \cos\sigma^1, \sin\sigma^1) \longmapsto (x^+(\sigma^+), x^-(\sigma^-)) \in M$, where

$$x^\pm(\sigma^\pm) := \tan\sigma^\pm/2 \tag{2.2}$$

and where $x^\pm := x^0 \pm x^1$ and $\sigma^\pm := \sigma^0 \pm \sigma^1$ denote light cone coordinates on M and on $\mathcal{M}$ respectively.

2.2. An infinite-dimensional group of spacetime symmetries

The case $d = 2$ is different from the higher dimensional case, since here the infinite-dimensional Fréchet nuclear Lie group

$$\mathcal{G} = (\widetilde{\mathrm{Diff}}_+(S^1) \times \widetilde{\mathrm{Diff}}_+(S^1))/\mathbb{Z} \tag{2.3}$$

acts as a group of spacetime symmetries on $\mathcal{M}$ (Schromerus and Mack 1990). To have a closer look at this action we recall that $\widetilde{\mathrm{Diff}}_+(S^1)$ is the universal covering group of the orientation-preserving diffeomorphisms of the circle. Its elements can be identified with smooth functions $\tau : \mathbb{R} \longmapsto \mathbb{R}$, obeying $\tau(\sigma + 2\pi) = \tau(\sigma) + 2\pi$ and $\tau' > 0$. With this identification a smooth action of $\widetilde{\mathrm{Diff}}_+(S^1) \times \widetilde{\mathrm{Diff}}_+(S^1)$ on $\mathcal{M}$ is defined in coordinates by $\sigma^\pm \longmapsto \tau^\pm(\sigma^\pm)$, where $(\tau^+, \tau^-) \in \widetilde{\mathrm{Diff}}_+(S^1) \times \widetilde{\mathrm{Diff}}_+(S^1)$. From (2.1) it follows that this action factorizes to an action of $\mathcal{G}$ on $\mathcal{M}$. The kernel $\mathcal{K}$ of this factorization is isomorphic to $\mathbb{Z}$ and consists of all (τ^+, τ^-) with $\tau^\pm(\sigma^\pm) = \sigma^\pm + 2\pi k^\pm$ and $k^+ + k^- = 0$. Elements in $\mathcal{G}$ are denoted by $[\tau^+, \tau^-]$, which indicates that $[\tau^+, \tau^-]$ is the image of (τ^+, τ^-) under the surjective homomorphism $\widetilde{\mathrm{Diff}}_+(S^1) \times \widetilde{\mathrm{Diff}}_+(S^1) \to \mathcal{G}$. Let $\tau \longmapsto [\tau]$ be the covering homomorphism $\widetilde{\mathrm{Diff}}_+(S^1) \to \mathrm{Diff}_+(S^1)$, which is given by $[\tau](\mathrm{e}^{\mathrm{i}\sigma}) := \mathrm{e}^{\mathrm{i}\tau(\sigma)}$ provided we identify S^1 with $\{z \in \mathbb{C} : |z| = 1\}$, and let

$$\Phi : \mathcal{G} \longrightarrow \mathrm{Diff}_+(\mathcal{S}^\infty) \times \mathrm{Diff}_+(\mathcal{S}^\infty) \tag{2.4}$$

be the homomorphism defined by $\Phi([\tau^+, \tau^-]) := ([\tau^+], [\tau^-])$, then $\ker \Phi = (\mathbb{Z} \times \mathbb{Z})/\mathcal{K}$ coincides with the centre $\mathcal{Z}$ of $\mathcal{G}$. It is illustrative to express these relations in terms of a commutative diagram:

$$\begin{array}{ccccc}
\mathcal{K} & \longrightarrow & \mathbb{Z} \times \mathbb{Z} & \longrightarrow & \mathcal{Z} \\
\| & & \downarrow & & \downarrow \\
\mathcal{K} & \longrightarrow & \widetilde{\mathrm{Diff}}_+(S^1) \times \widetilde{\mathrm{Diff}}_+(S^1) & \longrightarrow & \mathcal{G} \\
 & & \downarrow & & \downarrow \Phi \\
 & & \mathrm{Diff}_+(S^1) \times \mathrm{Diff}_+(S^1) & = & \mathrm{Diff}_+(S^1) \times \mathrm{Diff}_+(S^1).
\end{array} \tag{2.5}$$

2.3. The conformal group

The six-dimensional subgroup $(\widetilde{\mathrm{PSL}}(2,\mathbb{R}) \times \widetilde{\mathrm{PSL}}(2,\mathbb{R}))/\mathbb{Z} \subset \mathcal{G}$ is an infinite sheeted covering group of the conformal group $\mathrm{SO}(2,2)/\mathbb{Z}_2$ in two dimensions. Since $\mathrm{PSL}(2,\mathbb{R}) \simeq \mathrm{SU}(1,1)/\mathbb{Z}_2$ by Cayley transform, since $\mathrm{PSL}(2,\mathbb{R}) \times \mathrm{PSL}(2,\mathbb{R}) \simeq \mathrm{SO}(2,2)/\mathbb{Z}_2$ and since elements $[\tau^+, \tau^-]$ in $(\widetilde{\mathrm{PSL}}(2,\mathbb{R}) \times \widetilde{\mathrm{PSL}}(2,\mathbb{R}))/\mathbb{Z}$ are characterized by $\Phi([\tau^+, \tau^-]) := ([\tau^+], [\tau^-])$, where

$$[\tau](\mathrm{e}^{\mathrm{i}\sigma}) \equiv \mathrm{e}^{\mathrm{i}\tau(\sigma)} = \frac{\alpha \mathrm{e}^{\mathrm{i}\sigma} + \beta}{\bar{\beta}\mathrm{e}^{\mathrm{i}\sigma} + \bar{\alpha}} \qquad \text{for } \begin{pmatrix} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{pmatrix} \in \mathrm{SU}(1,1) \tag{2.6}$$

we have exactness of

$$\mathcal{Z} \longrightarrow (\widetilde{\mathrm{PSL}}(2,\mathbb{R}) \times \widetilde{\mathrm{PSL}}(2,\mathbb{R}))/\mathbb{Z} \longrightarrow \mathrm{SO}(2,2)/\mathbb{Z}_2. \tag{2.7}$$

The conformal group plays a special role not only because its generators have a direct relationship to physical observable quantities, but also because it turns out to be exactly the invariance group of the ground state of the second quantized free field theory which we shall introduce in the next section. In order to obtain the fundamental fields of the $(\widetilde{\mathrm{PSL}}(2,\mathbb{R}) \times \widetilde{\mathrm{PSL}}(2,\mathbb{R}))/\mathbb{Z}$ action on $\mathcal{M}$, we observe that (2.7) implies the existence of a unique lift of the one-parameter group

$(\exp tX^+, \exp tX^-)$ to $(\widetilde{\mathrm{PSL}}(2,\mathbb{R}) \times \widetilde{\mathrm{PSL}}(2,\mathbb{R}))/\mathbb{Z}$, where $X = (X^+, X^-)$ denotes a generator in $\mathfrak{su}(1,1) \oplus \mathfrak{su}(1,1) \simeq \mathfrak{so}(2,2)$. Since $\mathrm{Lie}((\widetilde{\mathrm{PSL}}(2,\mathbb{R}) \times \widetilde{\mathrm{PSL}}(2,\mathbb{R}))/\mathbb{Z}) = \mathrm{Lie}(\widetilde{\mathrm{PSL}}(2,\mathbb{R}) \times \widetilde{\mathrm{PSL}}(2,\mathbb{R})) \subset \mathrm{Lie}(\widetilde{\mathrm{Diff}}_+(S^1) \times \widetilde{\mathrm{Diff}}_+(S^1)) = \mathrm{Lie}(\mathrm{Diff}_+(S^1) \times \mathrm{Diff}_+(S^1)) = \mathrm{Vect}(S^1) \oplus \mathrm{Vect}(S^1)$ the lift of the previously mentioned one-parameter group uniquely determines vector fields $(\zeta_{X^+}, \zeta_{X^-})$, which we shall identify with smooth real-valued 2π periodic functions on $\mathbb{R}$. Then the flow $t \to \mathrm{Fl}_t^{\zeta_X}$ of ζ_X is a one-parameter group in $\widetilde{\mathrm{Diff}}_+(S^1)$ and, by construction, we have

$$\Phi([\mathrm{Fl}_t^{\zeta_{X^+}}, \mathrm{Fl}_t^{\zeta_{X^-}}]) = (\mathrm{L}_{\exp tX^+}, \mathrm{L}_{\exp tX^-}) \tag{2.8}$$

where Φ has been defined in (2.4). Here $A \mapsto \mathrm{L}_A$ is the homomorphism $\mathrm{SU}(1,1) \to \mathrm{Diff}_+(S^1)$, which has kernel $\{\mathbb{1}, -\mathbb{1}\}$, and which is defined for $A = \left(\begin{smallmatrix} \alpha & \beta \\ \bar\beta & \bar\alpha \end{smallmatrix}\right)$ by $\mathrm{L}_A(\mathrm{e}^{\mathrm{i}\sigma}) := (\alpha \mathrm{e}^{\mathrm{i}\sigma} + \beta)/(\bar\beta \mathrm{e}^{\mathrm{i}\sigma} + \bar\alpha)$ as in (2.6). Differentiation of the action of the one-parameter group $t \mapsto [\mathrm{Fl}_t^{\zeta_{X^+}}, \mathrm{Fl}_t^{\zeta_{X^-}}]$ of diffeomorphisms yields the fundamental field ς_X on $\mathrm{T}\mathcal{M}$. Note that $\varsigma_X \equiv \varsigma_{(X^+,X^-)} \equiv \varsigma_{X^+} + \varsigma_{X^-}$. Since $\mathcal{M} = \mathbb{R} \times S^1$ is a product of one-dimensional manifolds it is parallelizable, and $\mathrm{T}\mathcal{M}$ has a global frame $\{\partial_0, \partial_1\}$. The global frame of light cone vector fields $\{\partial_+, \partial_-\}$ is then obtained by $\partial_\pm := \frac{1}{2}(\partial_0 \pm \partial_1)$, and it is appropriate to re-express the fundamental fields of the action of the covering of the conformal group in terms of these fields. As can easily be seen

$$\varsigma_{X^\pm} = c^\pm \partial_\pm \tag{2.9}$$

where $c^\pm : \mathcal{M} \to \mathbb{R}$ are functions that depend only on one light cone coordinate, such that $c^\pm/\sigma^\mp = 0$. If $c^\pm =$ constant, then $c^+\partial_+ + c^-\partial_-$ generates the Abelian subgroup $(\widetilde{\mathrm{U}(1)} \times \widetilde{\mathrm{U}(1)})/\mathbb{Z}$ of translations $\sigma^\pm \longmapsto \sigma^\pm + c^\pm$, which is a covering group of the maximal torus $\mathbb{T}^2 \simeq \mathrm{U}(1) \times \mathrm{U}(1)$ in $\mathrm{SO}(2,2)/\mathbb{Z}_2$. By (2.2) we observe that the action of $(\widetilde{\mathrm{U}(1)} \times \widetilde{\mathrm{U}(1)})/\mathbb{Z}$ does not project to spacetime translations on Minkowski space. Nevertheless, as mentioned before, a conformally invariant and hence masseless quantum field theory exists on $\mathcal{M}$, and therefore we shall identify the generators of the unitary representation of $(\widetilde{\mathrm{U}(1)} \times \widetilde{\mathrm{U}(1)})/\mathbb{Z}$ on the Hilbert space of physical states with the energy–momentum observables of the theory. Then the vanishing mass of the particles implies that the most natural kinematical description is given in terms of light cone coordinates, and for that reason we shall focus our attention on the fundamental fields $\partial_\pm$ and on the one-parameter groups they generate.

2.4. Arbitrary one-parameter groups and the compact picture

Arbitrary one-parameter groups in $\mathcal{G}$ are generated by vector fields $(\xi^+, \xi^-) \in \mathrm{Vect}(S^1) \oplus \mathrm{Vect}(S^1)$. They are of the form $t \mapsto [\mathrm{Fl}_t^{\xi^+}, \mathrm{Fl}_t^{\xi^-}]$ for $t \in \mathbb{R}$. Note that by compactness of S^1 any $\xi \in \mathrm{Vect}(S^1)$ is complete, and the corresponding one-parameter group of orientation-preserving diffeomorphisms of the circle has a unique lift to its universal covering group $\widetilde{\mathrm{Diff}}_+(S^1)$. It is clear that $\xi \mapsto \mathrm{Fl}_1^{\xi}$ yields the exponential map $\exp : \mathrm{Vect}(S^1) \longrightarrow \widetilde{\mathrm{Diff}}_+(S^1)$. However care has to be taken, when working with the exponential, since $\exp$ is neither locally surjective nor globally injective (Pressley and Segal 1986) and its image generates a nowhere dense subgroup in $\widetilde{\mathrm{Diff}}_+(S^1)$ (Goodman and Wallach 1984).

A pair (p_+, p_-) of submersions

$$p_\pm : \mathcal{M} \longrightarrow S^1 \tag{2.10}$$

is defined by $p_\pm((\sigma^0, \cos\sigma^1, \sin\sigma^1)) := \mathrm{e}^{\mathrm{i}\sigma^\pm}$. In the following we shall tacitly identify the manifold S^1 with the complex unit circle. Moreover we shall refer to $\mathrm{e}^{\mathrm{i}\sigma^\pm} = z^\pm(\sigma^\pm)$ as compact picture coordinates. By definition of $p_\pm$ we observe

$$p_\pm \circ [\tau^+, \tau^-] = [\tau^\pm] \circ p_\pm \tag{2.11}$$

where $[\tau^+, \tau^-]$ and $[\tau^\pm]$ are regarded as diffeomorphisms acting on $\mathcal{M}$ and on S^1 respectively. As a consequence the fundamental fields $\varsigma_{X^\pm}$ of the $(\widetilde{\mathrm{PSL}}(2,\mathbb{R}) \times \widetilde{\mathrm{PSL}}(2,\mathbb{R}))/\mathbb{Z}$ action and the $\mathrm{SU}(1,1)/\mathbb{Z}_2$ action on S^1 are $p_\pm$-related

$$\mathrm{T}p_\pm \circ \varsigma_{X^\pm} = \zeta_{X^\pm} \circ p_\pm . \tag{2.12}$$

This terminates our brief overview of conformal kinematics in $1+1$ dimensions.

3. The free Majorana field

3.1. A Lagrangian scenario

We introduce the classical field theory of a single massless fermionic field Ψ on $(\mathcal{M}, g)$ which has Lagrangian

$$\mathcal{L} = \frac{\mathrm{i}}{2}\overline{\Psi}\gamma^\mu \overleftrightarrow{\partial}_\mu \Psi . \tag{3.1}$$

Here $\gamma^0 = \left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)$, $\gamma^1 = \left(\begin{smallmatrix}0&1\\-1&0\end{smallmatrix}\right)$ and $\gamma^5 = \gamma^0\gamma^1 = \left(\begin{smallmatrix}-1&0\\0&1\end{smallmatrix}\right)$ is a real representation of the γ-matrices in $1+1$ dimensions obeying $\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}$ where $\eta^{\mu\nu} = \mathrm{diag}(1,-1)$. The fermionic field Ψ is a smooth section of the trivial complex bundle $\mathcal{M} \times \mathbb{C}^2$, and its components $\left(\begin{smallmatrix}\psi_+\\\psi_-\end{smallmatrix}\right)$ of Ψ are chosen such that $\gamma^5\psi_+ = -\psi_+$ and $\gamma^5\psi_- = \psi_-$. Moreover $\overline{\Psi} := (C\Psi)^t\gamma^0$, where $\mathcal{C} : \mathcal{M} \times \mathbb{C}^2 \to \mathcal{M} \times \mathbb{C}^2$ is a fibre-preserving map which acts on each fibre as an antilinear involution, and which is given by

$$\mathcal{C}((\sigma^0, \cos\sigma^1, \sin\sigma^1, q^-, q^+)) := (\sigma^0, \cos\sigma^1, \sin\sigma^1, \mathrm{e}^{\mathrm{i}(\sigma^0-\sigma^1)}\overline{q^+}, \mathrm{e}^{\mathrm{i}(\sigma^0+\sigma^1)}\overline{q^-}). \tag{3.2}$$

Here the bar over $q^\pm$ denotes complex conjugation in $\mathbb{C}$. Integration of (3.1) over $\mathcal{M}$ yields the action

$$S[\Psi] = \int_{\mathcal{M}} \frac{\mathrm{i}}{2}\overline{\Psi}\gamma^\mu \overleftrightarrow{\partial}_\mu \Psi . \tag{3.3}$$

The functional $\Psi \longmapsto S[\Psi]$ is minimized by field configurations Ψ which satisfy the zero mass Dirac equation

$$\gamma^\mu\partial_\mu\Psi = 0$$

$$(\partial_\mu\overline{\Psi})\gamma^\mu = 0 .$$

Using the equations of motion, the on-shell expression of the energy–momentum tensor reads

$$\Theta^{\mu\nu}(\Psi) = \frac{\mathrm{i}}{2}\overline{\Psi}\gamma^{\mu}\overleftrightarrow{\partial}^{\nu}\Psi. \tag{3.4}$$

Neutral particles are described by a Majorana field which is obtained by imposing the reality condition $\mathcal{C}\circ\Psi = \Psi$ on field configurations satisfying (3.4). This implies that $\Psi = \binom{\psi_+}{\psi_-}$ becomes real in the sense of the bundle map $\mathcal{C}$. Since the components of the energy–momentum tensor correspond in principle to physical observable quantities, the densities must be periodic in light cone coordinates which forces the spinor fields to be either periodic or antiperiodic in $\sigma^{\pm}$. Here periodic (= Ramond) and antiperiodic (= Neuveu–Schwarz) boundary conditions are closely related to the two real line bundle structures on $\mathcal{M}$. We shall focus our attention on the latter, since they will lead naturally to the correlation functions of the conformal field theory of the two-dimensional Ising model (McCoy and Wu 1973).

3.2. *The geometric setting*

In order to establish a one-to-one correspondence between the chiral components of the Majorana field configurations and smooth sections in the Möbius strip, we observe that the reality condition $\mathcal{C}\circ\Psi = \Psi$ suggests regarding the Majorana field as a smooth section $\Psi : \mathcal{M} \to \mathcal{E}$, where $\mathcal{E}$ is the real sub-bundle in $\mathcal{M}\times\mathbb{C}^2$ with fibre $\mathbb{R}^2$, which is left invariant by $\mathcal{C}$. Then points in $\mathcal{E}$ can be parametrized by

$$(\sigma^0,\sigma^1,r^+,r^-) \longmapsto (\sigma^0,\cos\sigma^1,\sin\sigma^1,\mathrm{e}^{-\mathrm{i}(\sigma^0-\sigma^1)/2}r^-,\mathrm{e}^{-\mathrm{i}(\sigma^0+\sigma^1)/2}r^+) \tag{3.6}$$

where $r^{\pm}\in\mathbb{R}$. For $I_1 := (-\pi,\pi)$, $I_2 := (0,2)$ we define open sets $U_j \subset \mathcal{E}$ as the image of $I_j\times\mathbb{R}^2$ under the map (3.6). According to that $u_j : U_j \to I_j\times\mathbb{R}^2$ is defined as the inverse of (3.6). Then $(U_j,u_j)_{j\in\{1,2\}}$ is an atlas for $\mathcal{E}$. Moreover $U_1\cap U_2$ is the disjoint union of two open sets $V_\pm$ such that $u_1(V_+) = u_2(V_+) = (0,\pi)\times\mathbb{R}^2$, $u_1(V_-) = (-\pi,0)\times\mathbb{R}^2$ and $u_2(V_-) = (\pi,2\pi)\times\mathbb{R}^2$. From this and from (3.6) the transition functions $\Phi_{12}^{\pm} : V_\pm \to GL(2,\mathbb{R})$ can be computed. They turn out to be locally constant with $\Phi_{12}^{\pm} = \pm\mathbb{1}$, and we have the following proposition.

Proposition 3.1. The real vector bundle $\mathcal{E}\to\mathcal{M}$ over $\mathcal{M}$ is the Withney sum $\mathcal{E} = p_-^*\mathfrak{M}\oplus p_+^*\mathfrak{M}$ of the pullback $p_\pm^*\mathfrak{M}$ of the Möbius strip $\mathfrak{M}$ induced by the pair $(\mathfrak{M},p_\pm)$, where $p_\pm$ is the submersion defined in (2.10).

Proof. For simplicity of notation we shall identify the total space of the Möbius strip with the real sub-bundle

$$\mathfrak{M} := \{(\mathrm{e}^{\mathrm{i}\sigma},\mathrm{e}^{-\mathrm{i}\sigma/2}r) : (\sigma,r)\in(-\pi,\pi]\times\mathbb{R}\}$$

in $S^1\times\mathbb{C}$. Then by definition of the submersion $p_\pm$ we have

$$p_\pm^*\mathfrak{M} = \{(\sigma^0,\cos\sigma^1,\sin\sigma^1,\mathrm{e}^{-\mathrm{i}(\sigma^0\pm\sigma^1)/2}r^\pm) : (\sigma^0,\sigma^1,r^\pm)\in(-\pi,\pi]\times\mathbb{R}\}$$

and because of (3.6), $\mathcal{E} = p_-^*\mathfrak{M}\oplus p_+^*\mathfrak{M}$ follows. □

Note that by proposition 3.1

$$
\begin{array}{ccc}
p_\pm^*\mathfrak{M} & \xrightarrow{\pi_\pm} & \mathfrak{M} \\
\downarrow & & \downarrow \\
\mathcal{M} & \xrightarrow{p_\pm} & S^1
\end{array}
\tag{3.7}
$$

commutes. Moreover proposition 3.1 allows the complete characterization of the Majorana field configurations which obey NS boundary conditions. We have

Proposition 3.2. A classical Majorana field configuration obeying NS boundary conditions is a smooth section $\Psi : \mathcal{M} \to \mathcal{E}$ such that its chiral components $\psi_\pm$ correspond uniquely to smooth sections $f_\pm : S^1 \to \mathfrak{M}$ via $\pi_\pm \circ \psi_\pm = f_\pm \circ p_\pm$.

Proof. We observe that for each smooth function $\varphi : \mathbb{R} \to \mathbb{R}$ with $\varphi(\sigma + 2\pi k) = (-)^k f(\sigma)$, $k \in \mathbb{Z}$, there is an associated smooth section $f : S^1 \to \mathfrak{M}$ with $f(\mathrm{e}^{\mathrm{i}\sigma}) := (\mathrm{e}^{\mathrm{i}\sigma}, \mathrm{e}^{-\mathrm{i}\sigma/2}\varphi(\sigma))$. This correspondence is one to one, and given f_+ and f_- we define

$$
\begin{aligned}
(\star)\ \Psi((\sigma^0, \cos\sigma^1, \sin\sigma^1)) \\
&:= (\sigma^0, \cos\sigma^1, \sin\sigma^1, \mathrm{e}^{-\mathrm{i}\sigma^-/2}\varphi_-(\sigma^-), \mathrm{e}^{-\mathrm{i}\sigma^+/2}\varphi_+(\sigma^+)) \\
&= (\sigma^0, \cos\sigma^1, \sin\sigma^1, \psi_-, \psi_+).
\end{aligned}
$$

Then $C \circ \Psi = \Psi, \pi_\pm \circ \psi_\pm = f_\pm \circ p_\pm$, and by construction Ψ solves (3.4), which is equivalent to $\psi_-/\sigma^+ = \psi_+/\sigma^- = 0$ in coordinates. Conversely any Majorana field configuration satisfying the NS boundary conditions can be written in the form $(\star)$ with uniquely defined functions $\varphi_\pm$. □

4. Single-particle theory

4.1. Chiral decomposition and the construction of a single-particle Hilbert space

The fact that Majorana field configurations decompose into two independent chiral field configurations $\psi_\pm$ which are in one-to-one correspondence with sections $f_\pm : S^1 \longrightarrow \mathfrak{M}$ together with the factorization of the action of the classical conformal group $(\widetilde{\mathrm{Diff}}_+(S^1) \times \widetilde{\mathrm{Diff}}_+(S^1))/\mathbb{Z}$ in light cone coordinates allows fermionic theory to be decomposed into a product of two chiral parts which can be treated independently.

In the following the suffix $\pm$ will be dropped and we restrict attention to only one chiral component ψ of Ψ and to only one component of the energy–momentum tensor $\Theta^{\mu\nu}$. Note also that we drop the suffix $\pm$ from the periodic coordinate σ which parametrizes the unit circle in the compact picture.

The Möbius strip, which has been introduced in the proof of proposition 3.1 is obtained as the real sub-bundle of $S^1 \times \mathbb{C}$ which is left invariant by the bundle map

$$
C : S^1 \times \mathbb{C} \to S^1 \times \mathbb{C} \tag{4.1}
$$

given by $C((e^{i\sigma}, q)) := (e^{i\sigma}, e^{-i\sigma}\overline{q})$. By construction the maps $\mathcal{C}$ and C are related by $\pi_\pm \circ \mathcal{C} = C \circ \pi_\pm$. Let $\Gamma(S^1, S^1 \times \mathbb{C})$ be the complex vector space of smooth sections $f : S^1 \to S^1 \times \mathbb{C}$, then C induces an antilinear involution on $\Gamma(S^1, S^1 \times \mathbb{C})$ which we denote by abuse of language with C, and which is defined by

$$f \mapsto Cf := C \circ f \tag{4.2}$$

Then the real subspace $\frac{1}{2}(\mathrm{id} + C)\Gamma(S^1, S^1 \times \mathbb{C})$ coincides with $\Gamma(S^1, \mathfrak{M})$, the real vector space of smooth sections $f : S^1 \to \mathfrak{M}$. The sesquilinear standard inner product on $\mathbb{C}$ induces a fibre metric $(\cdot,\cdot) : \Gamma(S^1, S^1 \times \mathbb{C}) \times \Gamma(S^1, S^1 \times \mathbb{C}) \to C^\infty(S^1, \mathbb{C})$ which is defined by $(f_1, f_2)_\sigma = \overline{f_1(\sigma)} f_2(\sigma)$. Then multiplication of the $C^\infty(S^1, \mathbb{C})$-function $\sigma \to \frac{1}{2}\pi (f_1, f_2)_\sigma$ with the 1-form $\mathrm{d}\sigma$ on the circle yields the 1-form $\frac{1}{2}\pi(f_1, f_2)\,\mathrm{d}\sigma$ which can be integrated over S^1. The factor $\frac{1}{2}\pi$ has been introduced for later convenience. Thus we have obtained a non-degenerate positive definite sesquilinear form $\langle\cdot,\cdot\rangle$ on $\Gamma(S^1, S^1 \times \mathbb{C})$ which is given explicitly by

$$\begin{aligned} f_1, f_2 \longmapsto \langle f_1, f_2 \rangle &= \int_{S^1} \frac{1}{2\pi} (f_1, f_2)_\sigma \,\mathrm{d}\sigma \\ &= \int_{-\pi}^{+\pi} \frac{\mathrm{d}\sigma}{2\pi} \overline{f_1(\sigma)} f_2(\sigma). \end{aligned} \tag{4.3}$$

Let $\mathcal{H}$ be the Hilbert space completion of $\Gamma(S^1, S^1 \times \mathbb{C})$, then $\mathcal{H} \simeq L^2(S^1, \mathrm{d}\sigma)$, and C extends to an antilinear involution on $\mathcal{H}$ which satisfies

$$\langle Cf_1, Cf_2 \rangle = \langle f_2, f_1 \rangle. \tag{4.4}$$

Naturally $\mathrm{Re}(\mathcal{H}) = \frac{1}{2}(\mathbb{1} + C)\mathcal{H}$ is just the real Hilbert space completion of $\Gamma(S^1, \mathfrak{M})$. By taking proposition 3.2 into account, we shall refer to $\mathrm{Re}(\mathcal{H})$ as the real Hilbert space of chiral field configurations, and from this it is clear that $\mathcal{H}$ can be regarded as the single-particle Hilbert space of left moving or right moving Majorana fermions respectively. $\mathcal{H}$ contains dense linear subspaces whose elements differ in their analytic properties and which will be of interest in what follows. Let $\{e_m\}_{m\in\mathbb{Z}}, e_m(\sigma) := e^{im\sigma}$ be the complete orthonormal system in $\mathcal{H}$, and let for each $n \in \mathbb{N}, \mathcal{H}_n := \mathrm{Span}\{e_k : -n \leqslant k \leqslant n-1\}$ be the finite-dimensional subspace of trigonometric polynomials of degree less than n, then

$$\mathcal{H}_{\mathrm{pol}} := \bigcup_{n\in\mathbb{N}} \mathcal{H}_n \tag{4.5}$$

is dense in $\mathcal{H}$ by the Stone–Weierstrass theorem. Moreover $\mathcal{H}_{\mathrm{pol}}$ is contained in the dense linear subspace $\mathcal{H}_{\mathrm{an}}$ of real analytic elements, where real analyticity refers to σ coordinates. Equivalently $f \in \mathcal{H}_{\mathrm{an}}$ if and only if $f(-\mathrm{i}\log z)$ admits a convergent Laurent expansion in some annulus $\supset S^1$. Finally we denote by $\mathcal{H}_{\mathrm{smooth}}$ the dense linear subspace of complex-valued C^∞ functions on the circle. Then the inclusions $\mathcal{H} \supset \mathcal{H}_{\mathrm{smooth}} \supset \mathcal{H}_{\mathrm{an}} \supset \mathcal{H}_{\mathrm{pol}}$ are valid.

4.2. A unitary action of the group of spacetime symmetries on the single-particle Hilbert space

Next we look for an action of $\widetilde{\mathrm{Diff}}_+(S^1)$ on $\mathcal{H}$ which is restricted to an orthogonal action on $\mathrm{Re}(\mathcal{H})$. In principle the requirement of unitarity (respectively orthogonality on $\mathrm{Re}(\mathcal{H})$) can be motivated in the Lagrangian formalism but we will not pursue this line of reasoning. Instead we give an expression for a twisted action of $\widetilde{\mathrm{Diff}}_+(S^1)$ on $\mathcal{H}$ which differs from the one given in Pressley and Segal (1986).

Definition 4.1. Let $\tau \in \widetilde{\mathrm{Diff}}_+(S^1)$ then for any $f \in \mathcal{H}$ a linear operator U_τ on $\mathcal{H}$ is defined by

$$(U_\tau f)(\sigma) := \mathrm{e}^{\mathrm{i}(\tau^{-1}(\sigma)-\sigma)/2} f \circ \tau^{-1}(\sigma)\sqrt{(\tau^{-1})'(\sigma)}$$

The phase factor $\mathrm{e}^{\mathrm{i}/2(\tau^{-1}(\sigma)-\sigma)}$ which establishes the U_τ-invariance of $\mathrm{Re}(\mathcal{H})$ can be interpreted as a 1-cocycle $\tau \mapsto \mathrm{e}^{\mathrm{i}/2(\tau^{-1}-\mathrm{id}_{S^1})}$ on $\widetilde{\mathrm{Diff}}_+(S^1)$ with values in $C^\infty(S^1, \mathbb{T})$, where $\mathbb{T}$ is the one-dimensional torus (Bien 1988).

Proposition 4.2. Let $\tau \longrightarrow U_\tau$ given as in definition 4.1 and let Ξ be the generator of the centre $\{\Xi^n : n \in \mathbb{Z}\}$ in $\widetilde{\mathrm{Diff}}_+(S^1)$, then

$$U : \widetilde{\mathrm{Diff}}_+(S^1) \longrightarrow \mathfrak{B}(\mathcal{H})$$

is a unitary representation of $\widetilde{\mathrm{Diff}}_+(S^1)$ which commutes with C and which has $\ker U = \{\Xi^{2k} : k \in \mathbb{Z}\}$.

Proof. The verification of the representation property is straightforward. To see $\ker U = \{\Xi^{2k} : k \in \mathbb{Z}\}$, we observe that if $\tau(\sigma) = \sigma + a$ then $(U_\tau f)(\sigma) = \mathrm{e}^{-\mathrm{i}a/2} \times f(\sigma - a)$ and for $a = 2\pi$ we have $\tau = \Xi$ with $U_\Xi f = -f$. Since U is a homomorphism the centre $\{\Xi^n : n \in \mathbb{Z}\}$ is mapped onto the group $\{\mathbb{1}, -\mathbb{1}\}$, which implies $\ker U = \{\Xi^n : n \in \mathbb{Z}\}/\mathbb{Z}_2$. Unitarity follows from definition 4.1 together with (4.3), and using $(Cf)(\sigma) = \mathrm{e}^{-\mathrm{i}\sigma}\overline{f(\sigma)}$, we obtain $(CU_\tau)(\sigma) = (U_\tau C)(\sigma)$, which proves the first part of the assertion. □

4.3. *Topological preliminaries*

In the sequel we will denote the twofold covering $\widetilde{\mathrm{Diff}}_+(S^1)/\ker U$ of $\mathrm{Diff}_+(S^1)$ by $\mathrm{Diff}_+^\sim(S^1)$. Note that $U_\Xi = -1$ reflects NS boundary conditions or equivalently the non-trivial bundle structure of $\mathfrak{M}$. In order to investigate the continuity properties of U, when $\mathfrak{B}(\mathcal{H})$ is given the norm and the strong operator topology, we recall that $\widetilde{\mathrm{Diff}}_+(S^1)$ is a simple Fréchet nuclear Lie group modelled on the Lie algebra $\mathrm{Vect}(S^1)$ of C^∞ vector fields on the circle (Pressley and Segal 1986). Since $\mathrm{Vect}(S^1)$ is a free module over the ring $C^\infty(S^1, \mathbb{R})$ with basis $\mathrm{d}/\mathrm{d}\sigma$, vector fields may be identified with elements in $C^\infty(S^1, \mathbb{R})$. The topology on $\widetilde{\mathrm{Diff}}_+(S^1)$ is the initial topology induced from the injection $\widetilde{\mathrm{Diff}}_+(S^1) \hookrightarrow C^\infty(S^1, \mathbb{R})$; $\tau \mapsto \hat{\tau}$, where $\hat{\tau} := \tau(\sigma) - \sigma$. This implies that convergence of a net of diffeomorphisms $(\tau_\alpha)_{\alpha \in A}$ to a diffeomorphism τ is equivalent to uniform convergence $\|\tau_\alpha^{(n)} - \tau^{(n)}\|_\infty \to 0$ of all derivatives $\tau_\alpha^{(n)} = \mathrm{d}^n \tau_\alpha / \mathrm{d}\sigma^n$. Here A is a directed set, and since the Fréchet topology is metrizable, it is enough to consider the case $A = \mathbb{N}$. The topology on $\mathrm{Diff}_+(S^1)$ and on $\mathrm{Diff}_+^\sim(S^1)$ is the quotiend topology induced by the covering homomorphism.

4.4. *Strong continuity of the representation U*

If $\mathfrak{B}(\mathcal{H})$ carries norm topology, then $\tau \mapsto U_\tau$ is continuous provided $\mathrm{Diff}_+^\sim(S^1)$ has discrete topology (Pressley and Segal 1986). This follows from the observation that for each diffeomorphism $[\tau] \in \mathrm{Diff}_+(S^1) \neq \mathrm{id}_{S^1}$ there exists an open interval

$I \subset S^1$ which is sent into the interior of its complement by $[\tau]$. Choosing a function f in $\mathcal{H}$ which has support in I then implies $\langle U_\tau f, f\rangle = 0$, from which we conclude $\|U_\tau - \mathbb{1}\| \geqslant \sqrt{2}$. If $[\tau] = \mathrm{id}_{S^1}$, but $\tau \notin \ker U$ then $U_\tau = -\mathbb{1}$ and $\|U_\tau - \mathbb{1}\| = 2$. Hence $\|U_{\tau_1} - U_{\tau_2}\| \geqslant \sqrt{2}$ whenever $\tau_1 \circ \tau_2^{-1} \notin \ker U$. On physical grounds this result is not surprising, since it is well known from elementary quantum mechanics that, in general, kinematical symmetries cannot be implemented norm continuously on Hilbert space. Therfore the strong operator topology on $\mathfrak{B}(\mathcal{H})$ seems more appropriate and in fact it turns out that in this case $\tau \to U_\tau$ is continuous. To prove this assertion we need the following lemma.

Lemma 4.3. Let $(\tau_n)_{n\in\mathbb{N}}, \tau \in \widetilde{\mathrm{Diff}}_+(S^1)$ with $\tau_n \to \tau$. Then the sequence $(U_{\tau_n})_{n\in\mathbb{N}}$ of unitaries converges strongly to U_τ.

Proof. Since taking the inverse in $\widetilde{\mathrm{Diff}}_+(S^1)$ is continuous, we have $\tau_k^{-1} \to \tau^{-1}$ which in turn implies $\|\tau_k^{-1} - \tau^{-1}\|_\infty \to 0$ and $\|(\tau_k^{-1})' - (\tau^{-1})'\|_\infty \to 0$. For all $m, n \in \mathbb{N}$ we observe

$$\begin{aligned}
|\langle e_m, (U_{\tau_k} - U_\tau) e_n\rangle| &= \Bigg| \int_{-\pi}^{+\pi} \frac{\mathrm{d}\sigma}{2\pi} \mathrm{e}^{-\mathrm{i}(m+\frac{1}{2})\sigma} \\
&\qquad \left(\mathrm{e}^{\mathrm{i}(n+\frac{1}{2})\tau_k^{-1}(\sigma)} \sqrt{(\tau_k^{-1})'(\sigma)} - \mathrm{e}^{\mathrm{i}(n+\frac{1}{2})\tau^{-1}(\sigma)} \sqrt{(\tau^{-1})'(\sigma)} \right) \Bigg| \\
&\leqslant \left\| \sqrt{(\tau_k^{-1})'} - \sqrt{(\tau^{-1})'} \right\|_\infty + \left\| \sqrt{(\tau^{-1})'} \right\|_\infty \left\| \mathrm{e}^{\mathrm{i}(n+\frac{1}{2})\tau_k^{-1}} - \mathrm{e}^{\mathrm{i}(n+\frac{1}{2})\tau^{-1}} \right\|_\infty \\
&= : (\star).
\end{aligned}$$

But then $\lim_{k\to\infty} \langle e_m, (U_{\tau_k} - U_\tau) e_n\rangle = 0$, since by continuity of the square root $\sqrt{\ } : \mathbb{R}^+ \longmapsto \mathbb{R}^+$ and by equicontinuity of continuous functions on compact sets, $\|\tau_k^{-1} - \tau^{-1}\|_\infty \to 0$ and $\|(\tau_k^{-1})' - (\tau^{-1})'\|_\infty \to 0$ imply that $(\star)$ can be made arbitrarily small. Weak convergence $U_{\tau_n} \rightharpoonup U_\tau$ follows by an $\epsilon/3$ argument from the fact that any $f \in \mathcal{H}$ can be approximated by an element $\sum_{j=1}^n \langle e_{m_j}, f\rangle e_{m_j} \in \mathcal{H}_{\mathrm{pol}}$, since $\mathcal{H}_{\mathrm{pol}}$ is dense in $\mathcal{H}$. This finishes the proof, since a net of unitaries which converges weakly always converges strongly provided the limiting operator is unitary. $\square$

Taking lemma 4.3 into account, the continuity of the covering homomorphism $\widetilde{\mathrm{Diff}}_+(S^1) \to \mathrm{Diff}_+^\sim(S^1)$ together with $\ker U = \{\Xi^{2k} : k \in \mathbb{Z}\}$ and lemma 4.2 imply the following theorem.

Theorem 4.4. $U : \mathrm{Diff}_+^\sim(S^1) \longrightarrow \mathfrak{B}(\mathcal{H})$ is a faithful strongly continuous unitary representation of the twofold covering group of $\mathrm{Diff}_+(S^1)$ into the unitary operators in $\mathfrak{B}(\mathcal{H})$ which commute with the antilinear involution C on $\mathcal{H}$.

4.5. A Lie algebra of differential operators

Next we consider one-parameter subgroups. If $\xi \in \mathrm{Vect}(S^1)$, then $(\mathrm{Fl}^\xi{}_t)_{t\in\mathbb{R}}$ is a one-parameter group of diffeomorphisms and by theorem 4.3 $U_{\mathrm{Fl}^\xi_t}$ is a strongly continuous one-parameter group in $\mathfrak{B}(\mathcal{H})$, which—by Stones' theorem—must be generated by a self-adjoint densely defined operator D_ξ. In fact

Proposition 4.5. To each vector field $\xi \in \mathrm{Vect}(S^1)$ there corresponds a densely defined self-adjoint operator D_ξ such that $U_{\mathrm{Fl}^\xi_t} = \mathrm{e}^{\mathrm{i}D_\xi t}$, and

$$D_\xi = \mathrm{i}\left(\xi\frac{\mathrm{d}}{\mathrm{d}\sigma} + \frac{1}{2}\xi'\right) - \frac{1}{2}\xi$$

is essentially self-adjoint on the common U invariant domain $\mathcal{H}_{\mathrm{smooth}}$.

Proof. Going back to definition 4.1 one sees at a glance that $U_\tau\mathcal{H}_{\mathrm{smooth}} \subset \mathcal{H}_{\mathrm{smooth}}$. Then any unitary one-parameter group $U_{\mathrm{Fl}^\xi_t}$ is fixed without ambiguity by matrix elements on $\mathcal{H}_{\mathrm{smooth}}$, which is then a domain of essential self-adjointness for the generator. If we fix $f \in \mathcal{H}_{\mathrm{smooth}}$,

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} (\mathrm{e}^{\mathrm{i}D_\xi t} f)(\sigma) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \mathrm{e}^{\mathrm{i}(\mathrm{Fl}^\xi_{-t}(\sigma)-\sigma)/2} f \circ \mathrm{Fl}^\xi_{-t}(\sigma)\sqrt{\left(\mathrm{Fl}^\xi_{-t}\right)'(\sigma)}$$

exists and equals $-\xi f' - \frac{1}{2}\xi' f - (i/2)\xi f$. To see this put $(\mathrm{e}^{\mathrm{i}D_\xi t} f)(\sigma) =: q_t(\sigma)$. Then $(q_t(\sigma) - q_0(\sigma))/t = q'_{\lambda(\sigma)t}(\sigma)$ by the mean value theorem, where $0 \leqslant \lambda(\sigma) \leqslant 1$ and by continuity properties of Fl^ξ_t and f we have $\|q'_t - q'_0\|_\infty \to 0$ as $t \to 0$. Hence

$$\lim_{t\searrow 0}\left\|\frac{U_{\mathrm{Fl}^\xi_t} f - f}{t} - (\mathrm{i}D_\xi f)\right\|_2^2 = \lim_{t\searrow 0}\int_{-\pi}^{+\pi}\frac{\mathrm{d}\sigma}{2\pi}|q'_{\lambda(\sigma)}(\sigma) - q'_0(\sigma)|^2 = 0. \qquad \square$$

It is clear that $\xi \mapsto D_\xi$ is a Lie algebra homomorphism, and $\mathrm{i}[D_\xi, D_\eta] = D_{[\xi,\eta]}$ is easily verified on $\mathcal{H}_{\mathrm{smooth}}$. Note that here $[\xi,\eta] = -(\xi\eta' - \eta\xi')$ denotes the Lie bracket in the Lie algebra of the diffeomorphism group. We recall that this bracket is minus the Lie bracket of vector fields. For the investigation of topological properties of this map we switch to the resolvent $(D_\xi - z)^{-1}$ in order to work with bounded operators only. We recall that a sequence of self-adjoint operators is said to converge in the strong resolvent sense if their resolvents converge in the strong operator topology. A sufficient condition for strong resolvent convergence is then given by pointwise convergence of the sequence of self-adjoints on a common domain of essential self-adjointness (Weidman 1980).

Proposition 4.6. The map $\mathrm{Vect}(S^1) \to \mathfrak{B}(\mathcal{H})$ given by $\xi \mapsto (D_\xi - z)^{-1}$ is strongly continuous for each $z \in \mathbb{C}\setminus\mathbb{R}$.

Proof. By linearity of $\xi \mapsto D_\xi$ it is enough to verify $\xi_n \to 0 \Rightarrow D_{\xi_n} \to 0$ on $\mathcal{H}_{\mathrm{smooth}}$. But this follows from

$$\begin{aligned}\|D_{\xi_n} f\|_2^2 &= \int_{-\pi}^{+\pi}\frac{\mathrm{d}\sigma}{2\pi}\left|\xi_n(\sigma)\left(\mathrm{i}f' - \frac{1}{2}f\right)(\sigma) + \frac{\mathrm{i}}{2}\xi'_n(\sigma)f(\sigma)\right|^2 \\ &\leqslant \|\xi_n\|_\infty^2 c_1(f) + \|\xi'_n\|_\infty^2 c_2(f) + 2\|\xi'_n\|_\infty\|\xi_n\|_\infty c_3(f)\end{aligned}$$

where $c_i(f)$ are real positive constants depending on f. $\square$

Note that the preceding proofs rely heavily on the smoothness of the objects involved. This allows, in principle, due to U invariance of $\mathrm{Re}(\mathcal{H}_{\mathrm{smooth}})$, U_τ to be reformulated in an entirely geometrical context as C^∞ map $\Gamma(S^1,\mathfrak{M}) \to \Gamma(S^1,\mathfrak{M})$. We shall, however, stick to the algebraic point of view which is the appropriate one when second quantization is concerned.

4.6. The compact picture Hamiltonian and its positive energy spectral projection

In fact the last ingredient we need for the second quantization process is the spectral projection P of the compact picture Hamiltonian on the positive energy subspace in $\mathcal{H}$. Since the light cone fields $\partial_\pm$ are $p_\pm$ related to $\mathrm{d}/\mathrm{d}\sigma$ by (2.12), we shall identify $D_{\mathrm{d}/\mathrm{d}\sigma}$ with the compact picture Hamiltonian. For ease of notation we denote the latter by D_0, and with $D_0 = \mathrm{i}(\partial/\partial\sigma) - \frac{1}{2}$ it follows that $\mathrm{Spec}(D_0) = \mathbb{Z} - \frac{1}{2}$ is entirely discrete, since D_0 has non-degenerate eigenvalues $-n - \frac{1}{2}$ corresponding to the eigenvectors e_n, which form a total set in $\mathcal{H}$. The positive energy subspace is then obtained as the closure of $\mathrm{Span}\{e_m : m < 0\}$ and the corresponding spectral projection P satisfies

$$CP = (\mathbb{1} - P)C \tag{4.6}$$

since $CD_0 = -D_0 C$. P induces a polarization $P\mathcal{H} \oplus (\mathbb{1} - P)\mathcal{H}$ on $\mathcal{H}$ such that by (3.13) C acts as an antilinear isometry from $P\mathcal{H}$ to $(\mathbb{1} - P)\mathcal{H}$ and *vice versa*. In σ respective z-coordinates $\mathbb{1} - P$ can be written as an integral operator with singular kernel K which turns out to be the chiral contribution of the two-point function of the free fermionic field theory.

Proposition 4.7. Let $((\mathbb{1} - P)f)(\sigma) = \int_{-\pi}^{\pi} \mathrm{d}\sigma' \, K(\sigma,\sigma') f(\sigma')$. Then

$$2\pi K(\sigma,\sigma') = \frac{\mathrm{e}^{\mathrm{i}(\sigma' - i0)}}{\mathrm{e}^{\mathrm{i}(\sigma' - i0)} - \mathrm{e}^{\mathrm{i}\sigma}}.$$

Proof. It is sufficient to verify

$$\lim_{\epsilon \searrow 0} \int_{-\pi}^{+\pi} \frac{\mathrm{d}\sigma'}{2\pi} \frac{\mathrm{e}^{\mathrm{i}(\sigma' - \mathrm{i}\epsilon)}}{\mathrm{e}^{\mathrm{i}(\sigma' - \mathrm{i}\epsilon)} - \mathrm{e}^{\mathrm{i}\sigma}} \mathrm{e}^{\mathrm{i}n\sigma'} \equiv \lim_{\epsilon \searrow 0} (\star\star) = \begin{cases} \mathrm{e}^{\mathrm{i}n\sigma} & \text{for } n \neq 0 \\ 0 & \text{for } n < 0. \end{cases}$$

But this follows from

$$(\star\star) = \oint_{|z| = \mathrm{e}^{\epsilon}} \frac{\mathrm{d}z}{2\pi\mathrm{i}} \frac{\mathrm{e}^{-n\epsilon} z^n}{z - \mathrm{e}^{\mathrm{i}\sigma}} = \begin{cases} \mathrm{e}^{-n\epsilon}\mathrm{e}^{\mathrm{i}n\sigma} & \text{if } n \geqslant 0 \\ 0 & \text{if } n < 0 \end{cases}$$

by Cauchy's integral formula. □

5. Quasifree second quantization

5.1. The algebraic level

Each pair $(\mathcal{H}, C)$, where $\mathcal{H}$ is a separable Hilbert space, and $C : \mathcal{H} \to \mathcal{H}$ is an antilinear involution, has associated with it a self-dual CAR algebra $\mathfrak{A}(\mathcal{H}, C)$ which is generated from the range of a continuous $\mathbb{C}$-linear map $\Psi : \mathcal{H} \to \mathfrak{A}(\mathcal{H}, C)$, obeying $\Psi(f)^* = \Psi(Cf), \Psi(f_1)^*\Psi(f_2) + \Psi(f_2)\Psi(f_1)^* = \langle f_1, f_2\rangle \mathbb{1}$ and $\|\Psi(f)\|^2 \leqslant \langle f, f\rangle$ (Araki 1987). As an UHF algebra, $\mathfrak{A}(\mathcal{H}, C)$ is obtained as an inductive limit of an ascending net of I_{2n} factors $\mathfrak{A}(\mathcal{H}_n, C_n) \simeq \mathfrak{B}(\mathbb{C}^{2^n})$. Here $C_n := C \restriction \mathcal{H}_n$, $\mathcal{H}_N := \mathrm{Span}\{e_k : -n \leqslant k \leqslant n - 1\}$. $\mathfrak{A}_{\mathrm{pol}}(\mathcal{H}, C) := \bigcup_{n \in \mathbb{N}} \mathfrak{A}(\mathcal{H}_n, C_n)$ denotes the norm dense subalgebra spanned by all finite monomials $\Psi(e_{n_1}) \ldots \Psi(e_{n_k})$. The unitary action of $\widetilde{\mathrm{Diff}}_+(S^1)$ on the one-particle Hilbert space $\mathcal{H}$ can now be lifted to an action on $\mathfrak{A}(\mathcal{H}, C)$. Let $\mathrm{Bog}(\mathcal{H}, C)$ be the subgroup of all unitaries in $\mathfrak{B}(\mathcal{H})$ which commute with C, then for each U the map $\Psi(f) \mapsto \Psi(Uf)$ extends to the Bogoliubov automorphism $\alpha_U \in \mathrm{Aut}(\mathfrak{A}(\mathcal{H}, C))$. By theorem 4.4 we have $U_\tau \in \mathrm{Bog}(\mathcal{H}, C)$ for each $\tau \in \widetilde{\mathrm{Diff}}_+(S^1)$, and $\tau \mapsto \alpha_\tau := \alpha_{U_\tau}$ is a faithful representation of $\mathrm{Diff}_+^\sim(S^1)$ into the automorphism group $\mathrm{Aut}(\mathfrak{A}(\mathcal{H}, C))$. Let $\mathfrak{B}(\mathfrak{A}(\mathcal{H}, C))$ be the Banach algebra of all bounded linear operators on the Banach space $\mathfrak{A}(\mathcal{H}, C)$, equipped with the strong operator topology, then $\mathrm{Aut}(\mathfrak{A}(\mathcal{H}, C))$ is a subgroup in $\mathfrak{B}(\mathfrak{A}(\mathcal{H}, C))$ which is given the initial topology from the imbedding $\mathrm{Aut}(\mathfrak{A}(\mathcal{H}, C)) \hookrightarrow \mathfrak{B}(\mathfrak{A}(\mathcal{H}, C))$. As a consequence of the boundedness of $\Psi : \mathcal{H} \to \mathfrak{A}(\mathcal{H}, C)$ together with lemma 4.3 and proposition 4.5 we obtain the following proposition.

Proposition 5.1.

(i) The injective homomorphism $\mathrm{Diff}_+^\sim(S^1)\tau \mapsto \alpha_\tau$ is continuous.

(ii) To each vector field $\xi \in \mathrm{Vect}(S^1)$ there corresponds a closed symmetric derivation δ_ξ on $\mathrm{dom}(\delta_\xi)$. Moreover $\bigcap_{\xi \in \mathrm{Vect}(S^1)} \mathrm{dom}(\delta_\xi) \supset \mathrm{Span}\{\Psi(f_1) \ldots \Psi(f_n) : n \in \mathbb{N}, f_j \in \mathcal{H}_{\mathrm{smooth}}\}$ is a common norm dense α-invariant $*$-subalgebra in $\mathfrak{A}(\mathcal{H}, C)$, on which

$$\delta_\xi(\Psi(f_1) \ldots \Psi(f_n)) = \sum_{j=1}^{n} \Psi(f_1) \ldots \Psi(\mathrm{i} D_\xi f_j) \ldots \Psi(f_n)$$

is valid.

Proof. (i) Let $(\tau_n)_{n\in\mathbb{N}}, \tau \in \widetilde{\mathrm{Diff}}_+(S^1)$ with $\tau_n \to \tau$. We observe $\alpha_{\tau_n}(\Psi(f)) \to \alpha_\tau(\Psi(f))$ in norm, since $\|\alpha_{\tau_n}(\Psi(f)) - \alpha_\tau(\Psi(f))\| = \|\Psi(U_{\tau_n} f - U_\tau f)\| \leqslant \|U_{\tau_n} f - U_\tau f\|$. Now let $X := \prod_{j=1}^{k} \Psi(f_j)$ be an arbitrary monomial in $\mathfrak{A}_{\mathrm{pol}}(\mathcal{H}, C)$, then we find

$$\left\| \alpha_{\tau_n} \left(\prod_{j=1}^{k} \Psi(f_j) \right) - \alpha_\tau \left(\prod_{j=1}^{k} \Psi(f_j) \right) \right\| \leqslant \sum_{j=1}^{k} M_j \|U_{\tau_k} f_j - U_\tau f_j\|$$

where $M_j := \|f_j\|^{-1} \prod_{l=1}^{k} \|f_l\|$, which implies $\alpha_{\tau_n}(X) \to \alpha_\tau(X)$. Since $\|\alpha_\tau\| = 1$, pointwise convergence of the net $(\alpha_{\tau_n})_{n\in\mathbb{N}}$ on all of $\mathfrak{A}(\mathcal{H}, C)$ follows from an $\epsilon/3$ argument, and by definition of the topology in $\mathrm{Aut}(\mathfrak{A}(\mathcal{H}, C))$ the map $\tau \to \alpha_\tau$ is continuous.

(ii) With (i) we conclude that $t \mapsto \alpha_{\mathrm{Fl}^\xi_t}$ is a strongly continuous one-parameter group in $\mathfrak{B}(\mathfrak{A}(\mathcal{H}, C))$, which possesses a densely defined unbounded closed generator δ_ξ (Miwa 1984). Unboundedness of the latter follows from the fact that $\tau \mapsto \alpha_{\mathrm{Fl}^\xi_t}$ is not uniformly continuous. From U invariance of $\mathcal{H}_{\mathrm{smooth}}$ we deduce that the norm dense $*$-subalgebra generated by the linear span of all monomials $\Psi(f_1) \ldots \Psi(f_n)$ remains invariant under the action of α. Then $\|t^{-1}(\alpha_{\mathrm{Fl}^\xi_t}(\Psi(f)) - \Psi(f)) - \Psi(D_\xi f)\| \leqslant \|t^{-1}(U_\tau f - f) - D_\xi f\|$ implies

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \alpha_{\mathrm{Fl}^\xi_t}(\Psi(f)) \overset{(\star)}{=} \Psi(\mathrm{i} D_\xi f)$$

and with $(\star)$ the Leibniz rule is easily verified. This proves that δ_ξ is a symmetric $*$-derivation on $\mathfrak{A}(\mathcal{H}, C)$, since in addition $\delta_\xi(X^*) = \delta_\xi(X)^*$. $\square$

5.2. *A Fock state over $\mathfrak{A}(\mathcal{H}, C)$*

Given an orthogonal projection P in $\mathcal{H}$ with $CP = (\mathbb{1} - P)C$, there is an associated quasifree pure Fock state ω_P over $\mathfrak{A}(\mathcal{H}, C)$ which is uniquely fixed by the requirement $Pf = 0 \Rightarrow \omega_P(\Psi(f)^*\Psi(f)) = 0$ (Araki 1987). Since for a quasifree state all truncated functionals for $n > 2$ vanish, expectation values of arbitrary monomials $\Psi(f_1)\ldots\Psi(f_n)$ are expressed as Pfaffians of antisymmetric matrices $A_{ij} = -A_{ji}$ which have entries $A_{ij} = \langle Cf_i, Pf_j\rangle$ for $i < j$. Therefore the whole information is contained in the two-point function

$$\omega_P(\Psi(f_1)\Psi(f_2)) = \langle Cf_1, Pf_2\rangle. \tag{5.1}$$

Let P be the projection on the subspace of positive energy of the compact picture Hamiltonian D_0, introduced in section 4, and let ω_P be the associated Fock state over $\mathfrak{A}(\mathcal{H}, C)$. Then for $f_1, f_2 \in \mathcal{H}_{\mathrm{an}}, z \mapsto \tilde{f}_i(z) := f_i(-\mathrm{i}\log z)$ is analytic in a neighbourhood of the complex unit circle, and together with proposition 4.7, the two-point function (5.1) can be rewritten on $\mathcal{H}_{\mathrm{an}}$ in terms of complex contour integrals

Proposition 5.2. Let $f_1, f_2 \in \mathcal{H}_{\mathrm{an}}$, then

$$\omega_P(\Psi(f_1)\Psi(f_2)) = \oint_{|z_1|>|z_2|} \frac{\mathrm{d}z_1}{2\pi\mathrm{i}} \oint_{|z_2|=1} \frac{\mathrm{d}z_2}{2\pi\mathrm{i}} \tilde{f}_1(z_1)\frac{1}{z_1 - z_2}\tilde{f}_2(z_2)$$

where the z_1 integration is performed first.

Proof. With $(Cf)(\sigma) = \mathrm{e}^{-\mathrm{i}\sigma}\overline{f(\sigma)}$ we have

$$\begin{aligned}
\langle Cf_1, Pf_2\rangle &= \langle C(\mathbb{1} - P)f_1, f_2\rangle \\
&= \oint_{|z_2|=1} \frac{\mathrm{d}z_2}{2\pi\mathrm{i}}((\mathbb{1} - P)\tilde{f}_1)(z_2)\tilde{f}_2(z_2) \\
&\overset{(\star)}{=} \oint_{|z_2|=1} \frac{\mathrm{d}z_2}{2\pi\mathrm{i}} \left(\lim_{\epsilon\searrow 0} \oint_{|z_1|=1} \frac{\mathrm{d}z_1}{2\pi\mathrm{i}} \tilde{f}_1(z_1)\frac{1}{\mathrm{e}^{\epsilon}z_1 - z_2}\right)\tilde{f}_2(z_2) \\
&\overset{(\star\star)}{=} \oint_{|z_2|=1} \frac{\mathrm{d}z_2}{2\pi\mathrm{i}} \left(\oint_{|z_1|>|z_2|} \frac{\mathrm{d}z_1}{2\pi i} \tilde{f}_1(z_1)\frac{1}{z_1 - z_2}\right)\tilde{f}_2(z_2).
\end{aligned}$$

Here $(\star)$ follows from the fact that, according to proposition 4.7, the integral kernel of $\mathbb{1} - P$ in z-coordinates is $(e^0 z_1 - z_2)^{-1}$, and $(\star\star)$ follows from

$$\lim_{\epsilon\searrow 0} \oint_{|z|=1} \frac{\mathrm{d}z}{2\pi\mathrm{i}} \frac{\tilde{f}(z)}{e^{\epsilon}z - w} = \oint_{|z|>|w|} \frac{\mathrm{d}z}{2\pi\mathrm{i}} \frac{\tilde{f}(z)}{z - w}$$

which holds for any $f \in \mathcal{H}_{\mathrm{an}}$, and which can be easily verified by inserting the Laurent expansion for $\tilde{f}$ in both sides of the foregoing expression. □

The ϵ prescription in the preceding proof, and in the proof of proposition 4.7, suggests that the integral kernel of the bilinear form $f_1, f_2 \longmapsto \omega_P(\Psi(f_1)\Psi(f_2))$ is obtained as the boundary value of the function $(z_1 - z_2)^{-1}$, which is analytic in $\mathbb{C}^2 \setminus \{(z_1, z_2) : z_1 = z_2\}$. The singularity in $(z_1 - z_2)^{-1}$ is approached from the interior of the domain $\{(z_1, z_2) : |z_1| > |z_2|\}$ which contains $S^1 \times S^1$ as a submanifold on its three-dimensional boundary $\{(z_1, z_2) : |z_1| = |z_2|\}$.

5.3. Conformal invariance of the ground state

Now the explicit form of the two-point function can be used to investigate the invariance property of the Fock state ω_P under the right action $\tau \mapsto \alpha_\tau^*(\omega_P) := \omega_P \circ \alpha_\tau$ of $\mathrm{Diff}_+^\sim(S^1)$ on the states over $\mathfrak{A}(\mathcal{H}, C)$. Note that this action leaves the subset of all Fock states invariant, since they are in one-to-one correspondence with orthogonal projections P satisfying $CP = (\mathbb{1} - P)C$, and $\alpha_\tau^*(\omega_P) = \omega_{U_\tau^* P U_\tau}$. In what follows it will sometimes be convenient to work in complex coordinates. From the discussion of conformal kinematics in section 2 and from definition 4.1, the $\mathrm{SU}(1,1)$ action on $\mathcal{H}$ in z-coordinates is given by

$$\tilde{f} \mapsto \widetilde{U_\tau f} \qquad \text{with} \qquad \widetilde{U_\tau f}(z) = \frac{1}{\alpha - \overline{\beta} z} \tilde{f}\left(\frac{\overline{\alpha} z - \beta}{-\overline{\beta} z + \alpha}\right) \tag{5.2}$$

where by (2.6) τ is related to an element $A = \begin{pmatrix} \alpha & \beta \\ \overline{\beta} & \overline{\alpha} \end{pmatrix} \in \mathrm{SU}(1,1)$ by $[\tau](z) = (\alpha z + \beta)/(\overline{\beta} z + \overline{\alpha})$. Note that the generator $-\mathbb{1}$ of the discrete $\mathbb{Z}_2$ subgroup in the centre of $\mathrm{Diff}_+^\sim(S^1)$ corresponds to the transformation $f \mapsto -f$. But since ω_P vanishes on odd monomials of the field operators, this transformation leaves the Fock state unchanged, and we may expect an effective action of $\mathrm{Diff}_+(S^1)$, instead of an action of its twofold covering. Moreover on physical grounds $\mathrm{SU}(1,1)$—or more precisely $\mathrm{SU}(1,1)/\mathbb{Z}_2$—invariance of the vacuum must be satisfied in a second quantized conformal field theory. This is equivalent to the requirement that the classical conformal group is unitarily implementable in the Fock representation associated to ω_P. In fact we have

Proposition 5.3. Let $A \in \mathrm{SU}(1,1)$, and let $[\tau](z) = (\alpha z + \beta)/(\overline{\beta} z + \overline{\alpha})$. Then $\alpha_\tau^*(\omega_P) = \omega_P$, which implies invariance of the ground state under the $\mathrm{SU}(1,1)$ action.

Proof. To prove proposition 5.3 it is enough to verify $U_\tau^* P U_\tau f = Pf$ for f in $\mathcal{H}_{\mathrm{an}}$, since $\mathcal{H}_{\mathrm{an}}$ is dense in $\mathcal{H}$. But this is equivalent to $U_\tau^*(\mathbb{1} - P)U_\tau f = (\mathbb{1} - P)f$, and we have

$$\begin{aligned}
(U_\tau^*(\mathbb{1} - P)U_\tau \tilde{f})(z) &= \sqrt{[\tau]'(z)} \oint_{|\zeta|>1} \frac{\mathrm{d}\zeta}{2\pi\mathrm{i}} \frac{1}{\zeta - [\tau](z)} \widetilde{U_\tau f}(\zeta) \text{ where } |[\tau](z)| = 1 \\
&= \sqrt{[\tau]'(z)} \oint_{|\zeta|>1} \frac{\mathrm{d}\zeta}{2\pi\mathrm{i}} \frac{1}{\zeta - [\tau](z)} \sqrt{([\tau]^{-1})'(\zeta)} \tilde{f} \circ [\tau]^{-1}(\zeta) \\
&\overset{(\star)}{=} \oint_{|[\tau](u)|>1} \frac{\mathrm{d}u}{2\pi\mathrm{i}} \sqrt{[\tau]'(u)} \frac{1}{[\tau](u) - [\tau](z)} \sqrt{[\tau]'(z)} \tilde{f}(u) \\
&= \oint_{|u|>|z|} \frac{\mathrm{d}u}{2\pi\mathrm{i}} \frac{1}{u - w} \tilde{f}(u) \\
&= ((\mathbb{1} - P)\tilde{f})(z).
\end{aligned}$$

We have put $u = [\tau]^{-1}(\zeta)$. Moreover $\mathrm{d}\zeta = [\tau](u)'\,\mathrm{d}u$ together with $(\sqrt{[\tau]'(u)})^{-1} = \sqrt{([\tau]^{-1})'(\zeta)}$ which follows from $\mathrm{d}/\mathrm{d}\zeta([\tau] \circ [\tau]^{-1}(\zeta)) = 1$ justifies $(\star)$. From the definition of $[\tau]$ from $[\tau]'(z) = (\overline{\beta} z + \overline{\alpha})^{-2}$ and from $|\alpha|^2 - |\beta|^2 = 1$ we deduce

$$\sqrt{[\tau]'(u)} \frac{1}{[\tau](u) - [\tau](z)} \sqrt{[\tau]'(z)} = \frac{1}{u - w}.$$

Furthermore $|[\tau]^{-1}(\zeta)| > 1$ leads to $|\zeta| > 1$ which is equivalent to $|u| > 1 \Rightarrow |[\tau](u)| > 1$. To see this note that $[\tau](z) = (\alpha z+\beta)/(\overline{\beta}z+\overline{\alpha})$ with $|\alpha|^2-|\beta|^2 = 1$. Then $z \mapsto [\tau](z)$ can be easily extended by analytic continuation to a bijection of the Riemann sphere $\widehat{\mathbb{C}}$ onto itself which leaves its equator $\{z \in \mathbb{C} : |z| = 1\} \simeq S^1$ invariant. With the notation $\widehat{\mathbb{C}}\backslash S^1 =: \mathrm{H}_{\{\infty\}} \cup \mathrm{H}_{\{0\}}$ where $\mathrm{H}_{\{0\}}$ and $\mathrm{H}_{\{\infty\}}$ denotes the hemisphere containing the south pole 0 and the north pole ∞ respectively we observe that by continuity of $z \mapsto [\tau](z)$ either $[\tau](\mathrm{H}_{\{\infty\}}) = \mathrm{H}_{\{0\}}$ or $[\tau](\mathrm{H}_{\{\infty\}}) = \mathrm{H}_{\{\infty\}}$ must be true, since $[\tau] : \widehat{\mathbb{C}}\backslash S^1 \longrightarrow \widehat{\mathbb{C}}\backslash S^1$ is still a bijection. But $z \mapsto [\tau](z)$ has a pole for $z = -\overline{\beta}/\overline{\alpha}$ with $|z|^2 = (1 + |\alpha|^2)/|\alpha|^2 > 1$ which implies $[\tau](\mathrm{H}_{\{\infty\}}) = \mathrm{H}_{\{\infty\}}$. Consequently any point in $\mathrm{H}_{\{\infty\}}$ is mapped into a point in $\mathrm{H}_{\{\infty\}}$ and therefore $|z| > 1 \Rightarrow |u| = |[\tau](z)| > 1$. □

6. Implementation on Fock space

6.1. Implementation of conformal symmetries

To summarize the discussion, we have obtained a triplet $\{\mathfrak{A}(\mathcal{H}, C), \alpha, \mathrm{Diff}^{\sim}_{+}(S^1)\}$, where α is a strongly continuous representation of $\mathrm{Diff}^{\sim}_{+}(S^1)$ on the self-dual CAR algebra in terms of Bogoliubov automorphisms, together with a Fock state ω_P which is $\mathrm{SU}(1,1)$ invariant, and which gives rise to the GNS triplet $(\pi_P, \mathcal{H}_P, \Omega_P)$. In order not to confuse the one-particle Hilbert space $\mathcal{H}$ with the representation space $\mathcal{H}_P$ of π_P we shall rename the latter by $\mathcal{F}(P\mathcal{H})$, which suggests that $\mathcal{H}_P = \mathcal{F}(P\mathcal{H})$ is just the Fock space of the positive energy subspace $P\mathcal{H}$. In addition we shall write $\Psi_P(f)$ instead of $\pi_P \circ \Psi(f)$.

It is clear that the automorphisms $\alpha_\tau, [\tau](\mathrm{e}^{\mathrm{i}\sigma}) = (\alpha \mathrm{e}^{\mathrm{i}\sigma} + \beta)/(\overline{\beta}\mathrm{e}^{\mathrm{i}\sigma} + \overline{\alpha})$ belonging to the previously mentioned $\mathrm{SU}(1,1)$ subgroup can be easily implemented on $\mathcal{F}(P\mathcal{H})$ by uniquely determined implementers $Q_P(U_\tau)$, which are given on the total set of finite particle vectors by

$$\begin{aligned} &Q_P(U_\tau)\Psi_P(f_1)\Psi_P(f_2)\ldots\Psi_P(f_n)\Omega_P \\ &\qquad = :\Psi_P(U_\tau f_1)\Psi_P(U_\tau f_2)\ldots\Psi_P(U_\tau f_n)\Omega_P. \end{aligned} \tag{6.1}$$

6.2. The general implementation problem

In general ω_P will not be left invariant by the action of $\mathrm{Diff}^{\sim}_{+}(S^1)$ and the question of implementability is more delicate. Since our goal is to obtain a unitary ray representation of the diffeomorphism group on Fock space, one has to show that the unitaries U_τ on $\mathcal{H}$ satisfy the Hilbert–Schmidt condition, which in fact is a necessary and sufficient condition for a Bogoliubov automorphism to be implemented. We recall that such an automorphism is said to be unitarily implementable if there exists an implementer $Q_P(U) \in \mathfrak{B}(\mathcal{F}(P\mathcal{H}))$ such that for any $A \in \mathfrak{A}(\mathcal{H}, \Gamma)$ the relation

$$Q_P(U)\pi_P(A)Q_P(U)^* = \pi_P \circ \alpha_U(A) \tag{6.2}$$

is valid. To treat the general case where $\alpha^*_\tau(\omega_P) \neq \omega_P$ we have to introduce the Current group $\mathrm{Curr}_P(\mathcal{H}, C)$ (Araki 1987, 1988), which is defined by

$$\begin{aligned} \mathrm{Curr}_P(\mathcal{H}, C) := \{&Q \in \mathfrak{B}(\mathcal{F}(P\mathcal{H})) : Q^* = Q^{-1} \text{ and} \\ &Q\Psi_P(f)Q^* = \Psi_P(U_Q f) \text{ for a } U_Q \in \mathrm{Bog}(\mathcal{H}, C)\}. \end{aligned} \tag{6.3}$$

By irreducibility of π_P the kernel of the map $U \longmapsto U_Q$ is isomorphic to the group $\mathrm{U}(1)$ and its range is the subgroup in $\mathrm{Bog}(\mathcal{H},\Gamma)$ which will be denoted by $\mathrm{O}_P(\mathcal{H},C)$. Then by definition of $\mathrm{Curr}_P(\mathcal{H},C)$ and $\mathrm{O}_P(\mathcal{H},C)$

$$\mathrm{U}(1) \longrightarrow \mathrm{Curr}_P(\mathcal{H},C) \longrightarrow \mathrm{O}_P(\mathcal{H},C) \tag{6.4}$$

is exact. Elements in $\mathrm{O}_P(\mathcal{H},C)$ are completely characterized by the Hilbert–Schmidt condition (Araki 1987), which states that if $U \in \mathrm{Bog}(\mathcal{H},C)$, then $U \in \mathrm{O}_P(\mathcal{H},C)$ if and only if $\|PU(\mathbb{1}-P)\|_{\mathrm{HS}} < \infty$ and $\|(\mathbb{1}-P)UP\|_{\mathrm{HS}} < \infty$. Note that $CPU(\mathbb{1}-P)C = (\mathbb{1}-P)UP$ implies automatically $\|PU(\mathbb{1}-P)\|_{\mathrm{HS}} = \|(\mathbb{1}-P)UP\|_{\mathrm{HS}}$, such that it is sufficient to prove one of the two conditions. Moreover by (6.3) we learn that α_U has an implementer if and only if $U \in \mathrm{O}_P(\mathcal{H},C)$. Therefore our next step is to show $U_\tau \in \mathrm{O}_P(\mathcal{H},C)$. We do this by a slight modification of a similar proof given in Segal (1981). First we need the following lemma.

Lemma 6.1. Let $\tau \in \widetilde{\mathrm{Diff}}_+(S^1)$ and let $n \geqslant 1, m \geqslant 0$. Then for each $k \in \mathbb{N}$ there exists a $C(k) \in \mathbb{R}^+$ such that $|\langle e_{-n}, U_\tau e_m\rangle| \leqslant C(k)/(m+n)^k$.

Proof. Put $\tau^{-1} =: \kappa$, then

$$\langle e_{-n}, U_\tau e_m\rangle = \int_{-\pi}^{+\pi} \frac{\mathrm{d}\sigma}{2\pi} \mathrm{e}^{\mathrm{i}(n\sigma + m\kappa(\sigma))} h(\sigma)$$

where $h(\sigma) =: \mathrm{e}^{\mathrm{i}(\kappa(\sigma)-\sigma)/2}\sqrt{\kappa'(\sigma)}$. For $t \in [0,1], \kappa_t(\sigma) := t\kappa(\sigma) + (1-t)\sigma$ is again in $\widetilde{\mathrm{Diff}}_+(S^1)$ due to $\kappa_t(\sigma+2\pi) = \kappa_t(\sigma)+2\pi$ and $\kappa_t' > 0$. If $t = m/(m+n)$ we have $\kappa_{m/(m+n)} = (n+m)(m/(m+n)\kappa(\sigma) + (1-m/(m+n))\sigma) = m\kappa(\sigma) + n\sigma$. But then

$$\begin{aligned}
\langle e_{-n}, U_\kappa e_m\rangle &= \int_{-\pi}^{+\pi} \frac{\mathrm{d}\sigma}{2\pi} \mathrm{e}^{\mathrm{i}(m+n)\kappa_{m/(m+n)}(\sigma)} h(\sigma) \\
&= \int_{-\pi}^{+\pi} \frac{\mathrm{d}\varphi}{2\pi} \mathrm{e}^{\mathrm{i}(m+n)\varphi} r_{m/(m+n)}(\varphi) \\
&= \int_{-\pi}^{+\pi} \frac{\mathrm{d}\varphi}{2\pi} \left(-\frac{\mathrm{i}}{m+n}\right)^k \left(\frac{\partial^k}{\partial\varphi^k} \mathrm{e}^{\mathrm{i}(m+n)\varphi}\right) r_{m/(m+n)}(\varphi) \\
&= \left(\frac{\mathrm{i}}{m+n}\right)^k \int_{-\pi}^{+\pi} \frac{\mathrm{d}\varphi}{2\pi} \mathrm{e}^{\mathrm{i}(m+n)\varphi} \frac{\partial^k}{\partial\varphi^k} r_{m/(m+n)}(\varphi).
\end{aligned}$$

Here $\varphi = r_{m/(m+n)}(\sigma)$ and $r_t(\varphi) =: h \circ \kappa_t^{-1}(\varphi)/\kappa_t' \circ \kappa_t^{-1}(\varphi)$. By construction of r_t, the function $(t,\varphi) \longmapsto (\partial^k/\partial\varphi^k) r_t(\varphi)$ is continuous in both t and φ and we obtain

$$|\langle e_{-n}, U_\tau e_m\rangle| \leqslant C(k)/(m+n)^k$$

where

$$C(k) =: \sup\left\{\left|\frac{\partial^k}{\partial\varphi^k} r_t(\varphi)\right| : t \in [0,1], \varphi \in [-\pi,\pi]\right\} < \infty$$

by the boundedness of continuous functions on compact sets. □

Proposition 6.2. Let $\tau \in \widetilde{\mathrm{Diff}}_+(S^1)$. Then $U_\tau \in \mathrm{O}_P(\mathcal{H}, C)$.

Proof. The assertion follows since the Hilbert–Schmidt condition is satisfied.

$$\begin{aligned}
\|PU_\tau(\mathbb{1}-P)\|^2_{\mathrm{HS}} &= \sum_{n=1}^{\infty}\sum_{m=0}^{\infty} \langle e_{-n}, U_\tau e_m\rangle\langle e_m, U_\tau e_{-n}\rangle \\
&= \sum_{n=1}^{\infty}\sum_{m=0}^{\infty} \frac{C(k)^2}{(n+m)^{2k}} \qquad \text{by the preceding lemma} \\
&= \sum_{u=1}^{\infty} \frac{1}{u^{2k}} \sum_{n+m=u} C(k)^2 \\
&= C(k)^2 \sum_{u=1}^{\infty} \frac{1}{u^{2k-1}} < \infty \qquad \text{for } k > 1.
\end{aligned}$$

□

6.3. *The explicit form of the implementers*

With respect to the polarization $P\mathcal{H} \oplus (\mathbb{1}-P)\mathcal{H}$ we rewrite U_τ as a operator-valued matrix

$$U_\tau = \begin{pmatrix} a(\tau) & b(\tau) \\ \overline{b}(\tau) & \overline{a}(\tau) \end{pmatrix} \tag{6.5}$$

where $a(\tau) := PU_\tau P, b(\tau) := PU_\tau(\mathbb{1}-P), \overline{a}(\tau) := Ca(\tau)C$ and $\overline{b}(\tau) := Cb(\tau)C$. By proposition 6.3 $b(\tau) : (\mathbb{1}-P)\mathcal{H} \to P\mathcal{H}$ is Hilbert–Schmidt, and $a(\tau)$ is Fredholm with vanishing index, since due to the antilinear involution $C \dim\ker a(\tau) = \dim\mathrm{co}\ker a(\tau)$. If U_τ is close enough to the identity, then $\|P - U_\tau^* P U_\tau\| < 1$, and no level crossing occurs, since $P\mathcal{H} \cap (\mathbb{1} - U_\tau^* P U_\tau)\mathcal{H} = \ker a(\tau) = \{0\}$. In this case $Q_P(U_\tau)$ can be easily defined as in (6.1) by its action on the total set of finite particle vectors

$$\begin{aligned}
&Q_P(U_\tau)\Psi_P(f_1)\ldots\Psi_P(f_n)\Omega_P \\
&\quad := \frac{e^{i\vartheta}}{\det(1 + A^*(U_\tau)A(U_\tau))^{1/4}} \\
&\qquad \times \Psi_P(U_\tau f_1)\ldots\Psi_P(U_\tau f_n)\exp\left(\frac{1}{2}c_P^* A(U)c_P^*\right)\Omega_P.
\end{aligned} \tag{6.6}$$

Here $A(U_\tau) := (\mathbb{1}-P)U_\tau(PU_\tau P)^{-1} \equiv \overline{b}(\tau)a(\tau)^{-1}$. Note that $a(\tau)^{-1}$ exists as a bounded inverse, since $a(\tau)$ is Fredholm. Existence of $\det(1 + A^*(U)A(U))$ follows from $\mathrm{Tr}(A^*(U_\tau)A(U_\tau)) < \infty$. In (6.6) we have introduced creation and annihilation operators c_P^*, c_P which are defined by $c_P(h) := \Psi_P(Ch)$ and $c_P^*(h) := \Psi_P(h)$ for $h \in P\mathcal{H}$. With this convention the quadratic form in the exponent is understood as $c_P^* A(U_\tau)c_P^* := \sum_{m,n=1}^{\infty} c_P^*(e_{-m})\langle Ce_{-m}, A(U_\tau)e_{-n}\rangle c_P^*(e_{-n})$. The general case $0 \neq \dim\ker a(\tau) < \infty$ can be traced back to the former. Pick an orthonormal base $\{g_1, \ldots, g_n\}$ in $\ker a(\tau)$. Then the operator $\prod_{j=1}^{n}(\Psi_P(g_j) + \Psi_P(Cg_j))$ implements a self-adjoint unitary $V_\tau \in \mathrm{O}_P(\mathcal{H}, C)$, such that $(-)^n V_\tau$ interchanges $\ker a(\tau) = \mathrm{Span}\{g_1, \ldots, g_n\}$ with $\ker\overline{a}(\tau) = \mathrm{Span}\{Cg_1, \ldots, Cg_n\}$ without any other change (Araki 1987, Ruijsenaars 1978). But then $U'_\tau := (-)^n U_\tau V_\tau \in \mathrm{O}_P(\mathcal{H}, C)$ and $\ker(-)^n PU_\tau V_\tau P = \{0\}$. Let $N =: \sum_{j=1}^{\infty} c_P^*(e_{-j})c_P(e_{-j})$ be the fermion number operator. Then $\exp(i\pi nN)$ implements the Bogoliubov automorphism $f \mapsto (-)^n f$ and, as a consequence, the unitary $Q_P(U'_\tau)\exp(i\pi nN)\prod_{j=1}^{n}(\Psi_P(g_j) + \Psi_P(Cg_j))$ is an implementer for U_τ, where $Q_P(U'_\tau)$ is given by (6.6).

6.4. A central extension of $Diff^{\sim}_{+}(S^1)$

We recall that Ξ is the generator of the centre in $\widetilde{\mathrm{Diff}}_+(S^1)$. Then $U_\Xi f = -f$ and $\exp(\mathrm{i}\pi N)\Psi_P(f)\exp(-\mathrm{i}\pi N) = \Psi_P(U_\Xi f)$, which shows that U_Ξ is implemented by $\exp(\mathrm{i}\pi N)$ multiplied by a phase factor. But $\exp(\mathrm{i}\pi N) = \Pi_{\mathrm{ev}} - \Pi_{\mathrm{odd}}$ commutes with all implementers, such that $Q_P(U_\tau)$ can be restricted to $\mathcal{F}_{\mathrm{ev}}(P\mathcal{H}) := \Pi_{\mathrm{ev}}\mathcal{F}(P\mathcal{H})$ and to $\mathcal{F}_{\mathrm{odd}}(P\mathcal{H}) := \Pi_{\mathrm{odd}}\mathcal{F}(P\mathcal{H})$ respectively. Here Π_{ev} and Π_{odd} are the orthogonal projections on the even, respectively odd, particle number subspaces in $\mathcal{F}(P\mathcal{H})$. With the benefit of hindsight we define the $\mathrm{Curr}_P(\mathcal{H}, C)$ subgroup $(\mathrm{Diff}^{\sim}_{+}(S^1))_{\mathrm{ext}}$ by

$$(\mathrm{Diff}^{\sim}_{+}(S^1))_{\mathrm{ext}} := \{Q \in \mathfrak{B}(\mathcal{F}(P\mathcal{H})) : Q^* = Q^{-1} \text{ and } \exists \tau \in \widetilde{\mathrm{Diff}}_+(S^1) \text{ such that } Q\Psi_P(f)Q^* = \Psi_P(U_\tau f) \forall f \in \mathcal{H}\}. \tag{6.7}$$

Since the range of U is isomorphic to $\widetilde{\mathrm{Diff}}_+(S^1)/\ker U \simeq \mathrm{Diff}^{\sim}_{+}(S^1)$, we shall tacitly identify elements in $\mathrm{Diff}^{\sim}_{+}(S^1)$ with unitaries U_τ. Then the exact sequence (6.3) implies that $(\mathrm{Diff}^{\sim}_{+}(S^1))_{\mathrm{ext}}$ is a central extension of $\mathrm{Diff}^{\sim}_{+}(S^1)$ by $\mathrm{U}(1)$. Moreover let $(\mathrm{Diff}_+(S^1))_{\mathrm{ext}}$ be the subgroup in $\mathfrak{B}(\mathcal{F}_{\mathrm{ev}}(P\mathcal{H}))$ which is obtained as the image of the homomorphism $Q_P(U_\tau) \mapsto \Pi_{\mathrm{ev}} Q_P(U_\tau)$, then the previously mentioned homomorphism has kernel $\{\mathbb{1}, \exp(\mathrm{i}\pi N)\} \simeq \mathbb{Z}_2$, and it follows that $(\mathrm{Diff}_+(S^1))_{\mathrm{ext}}$ is a central extension of the diffeomorphism group of the circle by $\mathrm{U}(1)$, and the following diagram commutes.

$$\begin{array}{ccccc} & & \mathbb{Z}_2 & = & \mathbb{Z}_2 \\ & & \downarrow & & \downarrow \\ \mathrm{U}(1) & \longrightarrow & (\mathrm{Diff}^{\sim}_{+}(S^1))_{\mathrm{ext}} & \longrightarrow & \mathrm{Diff}^{\sim}_{+}(S^1) \\ \| & & \downarrow & & \downarrow \\ \mathrm{U}(1) & \longrightarrow & (\mathrm{Diff}_{+}(S^1))_{\mathrm{ext}} & \longrightarrow & \mathrm{Diff}_{+}(S^1). \end{array} \tag{6.8}$$

6.5. A two-cocycle on $Diff_+(S^1)$

Let $\mathcal{U} \subset \mathrm{Diff}^{\sim}_{+}(S^1)$ be a neighbourhood of the identity, consisting of all U_τ with $\ker a(\tau) = 0$. Then an algebraic cross section $\mathcal{U} \to (\mathrm{Diff}^{\sim}_{+}(S^1))_{\mathrm{ext}}$, $U_\tau \mapsto \widehat{Q}_P(U_\tau)$ can be defined where $\widehat{Q}_P(U_\tau)$ is given by (6.6) and where in addition the phase $\mathrm{e}^{\mathrm{i}\vartheta}$ in (6.6) has been put equal to one. This choice of phase then implies the following lemma.

Lemma 6.3. For all $U_\tau \in \mathcal{U}$ let $\widehat{Q}_P(U_\tau)$ be defined as in (6.6) with $\mathrm{e}^{\mathrm{i}\vartheta} = 1$. Then

(i) $\quad U_{\tau_1}U_{\tau_2} = \mathbb{1} \Rightarrow \widehat{Q}_P(U_{\tau_1})\widehat{Q}_P(U_{\tau_2}) = \widehat{Q}_P(\mathbb{1}) = \mathbb{1}$

(ii) $\quad U_{\tau_1}U_{\tau_2} = -\mathbb{1} \Rightarrow \widehat{Q}_P(U_{\tau_1})\widehat{Q}_P(U_{\tau_2}) = \widehat{Q}_P(-\mathbb{1}) = \exp(\mathrm{i}\pi N)$.

Proof. Since $\widehat{Q}_P(U_\tau^*)$ and $\widehat{Q}_P(U_\tau)^*$ implement the same algebra automorphism, we must have $\widehat{Q}_P(U_\tau^*) = \mathrm{e}^{\mathrm{i}\varphi}\widehat{Q}_P(U_\tau)^*$. But by (6.6) both $\langle\Omega_P, \widehat{Q}_P(U_\tau)^*\Omega_P\rangle = \langle\widehat{Q}_P(U_\tau)\Omega_P, \Omega_P\rangle$ and $\langle\Omega_P, \widehat{Q}_P(U_\tau^*)\Omega_P\rangle$ are real positive numbers $\neq 0$, which implies $\mathrm{e}^{\mathrm{i}\varphi} = 1$. This proves (i). (ii) is obvious from (6.6). $\square$

The previously defined local cross section can always be extended to all of $\mathrm{Diff}^{\sim}_{+}(S^1)$ such that properties (i) and (ii) in lemma 6.3 are satisfied globally. Denoting the extension of this local cross section with the same symbol $U_\tau \mapsto \widehat{Q}_P(U_\tau)$ we then obtain a global two-cocycle $(U_{\tau_1}, U_{\tau_2}) \mapsto \lambda(U_{\tau_1}, U_{\tau_2})$ on $\mathrm{Diff}^{\sim}_{+}(S^1)$ from this extension, and

$$\widehat{Q}_P(U_{\tau_1})\widehat{Q}_P(U_{\tau_2}) = \lambda(U_{\tau_1}, U_{\tau_2})\widehat{Q}_P(U_{\tau_1}U_{\tau_2}). \tag{4.12}$$

Moreover by property (ii) we have $\lambda(U_{\tau_1}U_{\Xi}, U_{\tau_2}) = \lambda(U_{\tau_1}, U_{\Xi}U_{\tau_2}) = \lambda(U_{\tau_1}, U_{\tau_2})$, which implies that λ factorizes to a two-cocycle on $\mathrm{Diff}_+(S^1)$. The latter can be rewritten as $\exp(\mathrm{i}\omega([\tau_1],[\tau_2])) := \lambda(U_{\tau_1}, U_{\tau_2})$, and in a neighbourhood of the identity in $\mathrm{Diff}_+(S^1)$, it is straightforward to compute this cocycle explicitly.

Proposition 6.4. Let $\mathcal{V}$ be the neighbourhood of $\mathbb{1}$ in $\mathrm{Diff}_+(S^1)$ which is closed under multiplication and for which $[\tau] \in \mathcal{V}$ implies $U_\tau \in \mathcal{U}$. Then

$$\exp(\mathrm{i}\omega([\tau_1],[\tau_2])) = \frac{\det(a(\tau_1)^{-1}a(\tau_1 \circ \tau_2)a(\tau_2)^{-1})^{1/4}}{\det(a(\tau_2^{-1})^{-1}a((\tau_1 \circ \tau_2)^{-1})a(\tau_1^{-1})^{-1})^{1/4}}$$

for all $[\tau_1],[\tau_2] \in \mathcal{V}$. Here the branch of $z \mapsto z^{1/4}$ is chosen such that $z \mapsto 1 \Rightarrow z^{1/4} \mapsto 1$, and all determinants are taken in $P\mathcal{H}$.

Proof. $[\tau_i] \in \mathcal{V}$ for $i \in \{1,2\}$ implies $\ker a(\tau) = \ker a(\tau^{-1}) = \{0\}$ for each $\tau \in \{\tau_1, \tau_2, \tau_1 \circ \tau_2\}$. Therefore the implementers $\widehat{Q}_P(U_\tau)$ are given by (6.6) with the additional convention $\mathrm{e}^{\mathrm{i}\vartheta} = 1$. Taking vacuum expectation values of both sides of the identity $\widehat{Q}_P(U_{\tau_1})\widehat{Q}_P(U_{\tau_2}) = \exp(\mathrm{i}\omega([\tau_1],[\tau_2]))\widehat{Q}_P(U_{\tau_1 \circ \tau_2})$ we obtain by using $\widehat{Q}_P(U^*_{\tau_1}) = \widehat{Q}_P(U_{\tau_1})^*$ the expression

$$\begin{aligned}
&\exp(\mathrm{i}\omega([\tau_1],[\tau_2])) \\
&\quad = \frac{\det(\mathbb{1} + A^*(U_{\tau_1 \circ \tau_2})A(U_{\tau_1 \circ \tau_2}))^{1/4}}{\det(\mathbb{1} + A^*(U_{\tau_1})A(U_{\tau_1}))^{1/4}\det(\mathbb{1} + A^*(U_{\tau_2})A(U_{\tau_2}))^{1/4}} \\
&\qquad \times \langle \exp\left(\tfrac{1}{2}c^*_P A(U^*_{\tau_1})c^*_P\right)\Omega_P, \exp\left(\tfrac{1}{2}c^*_P A(U_{\tau_2})c^*_P\right)\Omega_P\rangle.
\end{aligned}$$

Since the ratio of the determinants is real, the first factor on the right-hand side does not contribute to the phase, and therefore it is sufficient to restrict attention to the contribution of the inner product only. Moreover $A(U_\tau) = (\mathbb{1} - P)U_\tau(PU_\tau P)^{-1}$ implies $\mathbb{1} + A^*(U^*_{\tau_1})A(U_{\tau_2}) = (PU_{\tau_1}P)^{-1}U_{\tau_1 \circ \tau_2}(PU_{\tau_2}P)^{-1} = a(\tau_1)^{-1}a(\tau_1 \circ \tau_2)a(\tau_2)^{-1} := X$. Then, using the CAR relations it is straightforward but tedious to compute

$$\begin{aligned}
&\langle \exp\left(\tfrac{1}{2}c^*_P A(U^*_{\tau_1})c^*_P\right)\Omega_P, \exp\left(\tfrac{1}{2}c^*_P A(U_{\tau_2})c^*_P\right)\Omega_P\rangle \\
&\qquad = \det(\mathbb{1} + A(U^*_{\tau_1})^* A(U_{\tau_2}))^{1/2} \\
&\qquad = \det(a(\tau_1)^{-1}a(\tau_1 \circ \tau_2)a(\tau_2)^{-1})^{1/2}.
\end{aligned}$$

Here the branch of the square root is taken where $z^{1/2} \to 1$ if $z \to 1$. Since $\det X^* = \overline{\det X}$ also holds in the infinite-dimensional case, the assertion follows from the observation $z/\overline{z} = \mathrm{e}^{2\mathrm{i}\arg(z)}$, and from $X^* = a(\tau_2^{-1})^{-1}a((\tau_1 \circ \tau_2)^{-1})a(\tau_1^{-1})^{-1}$. Then the cocyle is equal to $(\det X/\det X^*)^{1/4}$, and the previously mentioned choice of the square root then implies that $z^{1/4} \to 1$ if $z \to 1$.

□

6.6. The $C^\star$ algebra of chiral observables

From the preceding discussion we learn, that the map $[\tau] \mapsto \widehat{\Gamma}([\tau])$, where $\widehat{\Gamma}([\tau]) := \Pi_{ev} \widehat{Q}_P(U_\tau)$, is a projective representation of $\mathrm{Diff}_+(S^1)$ into the unitary operators in $\mathfrak{B}(\mathcal{F}_{ev}(P\mathcal{H}))$. Moreover the smallest C^* algebra in $\mathfrak{B}(\mathcal{F}_{ev}(P\mathcal{H}))$, which is generated by the range of the map introduced earlier, is identified with the abstract C^* algebra $\mathcal{O}$ of chiral observables. Hence $\mathcal{O}$ is the group algebra of the central extension $(\mathrm{Diff}_+(S^1))_{ext}$ of the diffeomorphism group of the circle. It is interesting to see that $\mathcal{O}$ can, in principle, be obtained from an abstract construction which parallels the well known construction of the Weyl algebra in quantum mechanics.

Theorem 6.5. Let $\mathcal{O}_{free}$ be the free complex vector space over the set of symbols $\{\mathcal{W}([\tau]) : [\tau] \in \mathrm{Diff}_+(S^1)\}$. Then $\mathcal{O}_{free}$ is given a $*$-algebra structure by

$$\mathcal{W}([\tau_1])\mathcal{W}([\tau_2]) = \exp(i\omega([\tau_1],[\tau_2]))\mathcal{W}([\tau_1 \circ \tau_2])$$
$$\mathcal{W}([\tau])^* = \mathcal{W}([\tau]^{-1}).$$

The linear functional ω_0 defined by

$$\omega_0(\mathcal{W}([\tau])) := \begin{cases} \det(a(\tau)a(\tau^{-1}))^{1/4} & \text{if } \ker a(\tau) = \{0\} \\ 0 & \text{if } \ker a(\tau) \neq \{0\} \end{cases}$$

is a state over $\mathcal{O}_{free}$, and if $(\pi_0, \mathcal{H}_0, \Omega_0)$ denotes the associated GNS triplet, then there exists a unitary $S : \mathcal{H}_0 \to \mathcal{F}_{ev}(P\mathcal{H})$ such that

$$S\pi_0(\mathcal{W}([\tau]))S^* = \widehat{\Gamma}([\tau]).$$

Proof. Positivity of the linear functional ω_0 follows from the fact that $\mathcal{W}([\tau]) \mapsto \widehat{\Gamma}([\tau])$ is a $*$-isomorphism, $\widehat{\Gamma}([\tau]) \mapsto \langle \Omega_P, \widehat{\Gamma}([\tau])\Omega_P\rangle$ is positive, and $\omega_0(\mathcal{W}([\tau])) \overset{(\star)}{=} \langle \Omega_P, \widehat{\Gamma}([\tau])\Omega_P\rangle$. Here $(\star)$ holds, since

$$\det(\mathbb{1} + A^*(U_\tau)A(U_\tau))^{-1/4} = \det(PU_\tau PU_\tau^* P)^{1/4} = \det(a(\tau)a(\tau^{-1}))^{1/4}$$

and since $\langle \Omega_P, \widehat{\Gamma}([\tau])\Omega_P\rangle = 0$ iff $\ker a(\tau) \neq \{0\}$. Hence ω_0 is a state over $\mathcal{O}_{free}$.

Since Ω_P is cyclic for $\mathcal{O}$, a unitary operator $S : \mathcal{H}_t \to \mathcal{F}_{ev}(\mathcal{P}\mathcal{H})$ is given by $S\pi_0(\mathcal{W}([\tau]))\Omega_0 := \widehat{\Gamma}([\tau])\Omega_P$. Then $\star$ implies unitarity and, by definition of S, we have $S\pi_0(\mathcal{W}([\tau]))S^* = \widehat{\Gamma}([\tau])$. □

7. Algebra of charges and Schwinger term

7.1. Implementation of one-parameter groups

In the preceding section we have only dealt with the implementation of fixed elements $U_\tau \in \mathrm{Diff}_+^\sim(S^1)$, which led to the construction of the central extension $(\mathrm{Diff}_+^\sim(S^1))_{ext}$ as a group of unitaries in $\mathfrak{B}(\mathcal{F}(P\mathcal{H}))$. Now we pose the question as to the conditions under which the strongly continuous one-parameter group $\{e^{iD_\xi t} : t \in \mathbb{R}\}$ can be lifted to a strongly continuous one-parameter group

$\{e^{i dQ_P(D_\xi)} : t \in \mathbb{R}\}$ in $(\mathrm{Diff}^{\sim}_{+}(S^1))_{\mathrm{ext}}$, such that $dQ_P(D_\xi)$ is self-adjoint on a domain containing the Fock vacuum, and $\langle \Omega_P, dQ_P(D_\xi)\rangle = 0$. It turns out, that the only condition which has to be satisfied for an affirmative answer is continuity of $\{e^{iD_\xi t} : t \in \mathbb{R}\}$ in the P-strong topology on $O_P(\mathcal{H}, C)$ (Araki 1987). Here the P-strong topology on $O_P(\mathcal{H}, C)$ is generated by the family of seminorms $\{\rho_f : f \in \mathcal{H}\}$ where $\rho_f(A) := \|Af\| + \|PA(\mathbb{1} - P)\|_{\mathrm{HS}}$, and it is easy to see that this topology is strictly finer than the strong operator topology. Moreover for the previously mentioned one-parameter group it is sufficient to prove $\|PD_\xi(\mathbb{1} - P)\|_{\mathrm{HS}} < \infty$, since this automatically entails P-strong continuity (Araki 1987). Note that due to $PD_\xi + D_\xi P = 0$ we have $\|PD_\xi(\mathbb{1} - P)\|_{\mathrm{HS}} = \|(\mathbb{1} - P)D_\xi P\|_{\mathrm{HS}}$. For a detailed treatment see Araki (1988) and Lundberg (1976).

Proposition 7.1. To each ξ in $\mathrm{Vect}(S^1)$, the one-parameter group $\{e^{iD_\xi} : t \in \mathbb{R}\}$ is continuous in the P-strong topology.

Proof. Let $\hat{\xi}(n)$ denote the nth Fourier mode of the C^∞ vector field ξ. Then

$$\begin{aligned}\langle e_m, D_\xi e_{-n}\rangle &= \int_{-\pi}^{+\pi} \frac{d\sigma}{2\pi} e^{-im\sigma}\left(i\xi\frac{d}{d\sigma} + \frac{i}{2}\xi' - \frac{1}{2}\xi\right) e^{-in\sigma} \\ &= \frac{1}{2}(n - m - 1)\int_{-\pi}^{+\pi} \frac{d\sigma}{2\pi} e^{-i(n+m)\sigma}\xi(\sigma) \\ &= \frac{1}{2}(n - m - 1)\hat{\xi}(n + m).\end{aligned}$$

Moreover

$$\begin{aligned}\|PD_\xi(\mathbb{1} - P)\|_{\mathrm{HS}} &= \sum_{n=1}^{\infty}\sum_{m=0}^{\infty}\langle e_m, D_\xi^* e_{-n}\rangle\langle e_{-n}, D_\xi e_m\rangle \\ &= \sum_{n=1}^{\infty}\sum_{m=0}^{\infty}\frac{1}{4}(n - m - 1)^2|\hat{\xi}(m + n)|^2 \\ &\leqslant \sum_{n=1}^{\infty}\sum_{m=0}^{\infty}(m + n)^2|\hat{\xi}(m + n)|^2 =: (\star).\end{aligned}$$

Now

$$\begin{aligned}(\star) &= \sum_{u=1}^{\infty} u^3|\hat{\xi}(u)|^2 \\ &\leqslant \sum_{u\in\mathbb{Z}} u^4|\hat{\xi}(u)|^2 \\ &= \sum_{u\in\mathbb{Z}}(-u^2\hat{\xi}(u))\overline{(-u^2\hat{\xi}(u))} \\ &= \langle \xi'', \xi''\rangle.\end{aligned}$$

But $\langle \xi'', \xi''\rangle \leqslant \infty$, since any derivative of a C^∞ function is square integrable, and with $u = n + m$ we have $\sum_{\substack{m+n=u \\ n\geqslant 1, m\geqslant 0}} 1 = u$. □

7.2. The Lie algebra of charges as a central extension of Vect(S^1)

Because $(\mathrm{Diff}^{\sim}_{+}(S^1))_{\mathrm{ext}}$ is a central extension of $\mathrm{Diff}^{\sim}_{+}(S^1)$ by U(1), its Lie algebra $\mathrm{Lie}(\mathrm{Diff}^{\sim}_{+}(S^1))_{\mathrm{ext}}$ must be isomorphic to a central extension $\mathrm{Vect}(S^1)\oplus\mathbb{R}$ of the Lie algebra of $\mathrm{Vect}(S^1)$ by $\mathbb{R}$. Here $\mathrm{Lie}(\mathrm{Diff}^{\sim}_{+}(S^1))_{\mathrm{ext}}$ is obtained as the algebra of unbounded self-adjoint generators of strongly continuous one-parameter groups in $(\mathrm{Diff}^{\sim}_{+}(S^1))_{\mathrm{ext}}$. We shall refer to $(\mathrm{Diff}^{\sim}_{+}(S^1))_{\mathrm{ext}}$ as the Lie algebra of charges, which can be justified from (7.4). In fact, proposition 7.1 establishes the existence of such a $\mathrm{dQ}_P(D_\xi)\in\mathrm{Lie}(\mathrm{Diff}^{\sim}_{+}(S^1))_{\mathrm{ext}}$ for each generator D_ξ, but in general the fact that $U_\tau\mapsto Q_P(U_\tau)$ is a projective representation of $\mathrm{Diff}^{\sim}_{+}(S^1)$ on $\mathcal{F}(P\mathcal{H})$ is reflected on the infinitesimal level by uniqueness of $\mathrm{dQ}_P(D_\xi)$ up to an additional constant. The situation is then summarized in

$$\mathbb{R}\longrightarrow\mathrm{Lie}(\mathrm{Diff}^{\sim}_{+}(S^1))_{\mathrm{ext}}\longrightarrow\mathrm{Vect}(S^1). \tag{7.1}$$

7.3. Normal ordering and Schwinger term

In the previous discussion this constant has been fixed by $\langle\Omega_P,\mathrm{dQ}_P(D_\xi)\Omega_P\rangle=0$, which comes from the normal ordering procedure of fermiomic currents, motivated from the requirement of positivity of the spectrum of the second quantized compact picture Hamiltonian D_0. But then $\xi\mapsto\mathrm{dQ}_P(D_\xi)$ is a cross section from $\mathrm{Vect}(S^1)$ into its central extension, and the Schwinger term $S(\xi,\eta)\mathbb{1}:=\mathrm{i}[\mathrm{dQ}_P(D_\xi),\mathrm{dQ}_P(D_\eta)]-\mathrm{dQ}_P(D_{[\xi,\eta]})$ measures the deviation of this cross section from being a Lie algebra homomorphism.

In Araki (1987) it has been proved that, given $H_i=H_i^*, CH_i+H_iC=0$ with the property $\|PH_i(\mathbb{1}-P)\|_{\mathrm{HS}}\leqslant\infty, i\in\{1,2\}$, and H_i not neccessarily bounded, the generators $\mathrm{dQ}_P(H_i)$ obey the relations

$$\mathrm{i}[\mathrm{dQ}_P(H_1),\mathrm{dQ}_P(H_2)]=\mathrm{dQ}_P(\mathrm{i}[H_1,H_2])+\frac{\mathrm{i}}{8}\,\mathrm{Tr}(F_P[F_P,H_1][F_P,H_2]). \tag{7.2}$$

Here $F_P:=2P-\mathbb{1}$ is a self-adjoint idempotent unitary which is diagonal on the polarization $P\mathcal{H}\oplus(\mathbb{1}-P)\mathcal{H}$. Moreover for H_i, the generator can be written as the formal expression

$$\mathrm{dQ}_P(H_i)=\tfrac{1}{2}\sum_{n,m}\langle e_n,H_ie_m\rangle:\Psi_P(e_n)\Psi_P(e_m)^*: \tag{7.3}$$

where the right-hand side converges when applied to the vacuum. Then with (7.2) it is easy to compute explicitly the Schwinger term $S(\cdot,\cdot)$.

Proposition 7.2. Let $\xi,\eta\in\mathrm{Vect}(S^1)$, then

$$S(\xi,\eta)=-\frac{1}{24}\int_{-\pi}^{+\pi}\frac{\mathrm{d}\sigma}{2\pi}(\xi'''+\xi')(\sigma)\eta(\sigma).$$

Proof. With $F_P=2P-\mathbb{1}$, $P=\sum_{n=1}^{\infty}e_{-n}\langle e_{-n},\cdot\rangle$, we find $\mathrm{Tr}(F_P[F_P,D_\xi][F_P,D_\eta])=\sum_{m\geqslant0}\sum_{n\geqslant0}\langle e_m,D_\xi e_{-n}\rangle\overline{\langle e_m,D_\eta e_{-n}\rangle}$. Then, using $\langle e_m,D_\xi e_{-n}\rangle=\frac{1}{2}(n-$

$m-1)\widehat{\xi}(n+m)$, we have

$$\begin{aligned}&\mathrm{Tr}(F_P[F_P,D_\xi][F_P,D_\eta])\\&\qquad=\sum_{m\geqslant 0}\sum_{n\geqslant 1}(n-m-1)^2(\widehat{\xi}(m+n)\overline{\widehat{\eta}(m+n)}-\overline{\widehat{\xi}(m+n)}\widehat{\eta}(m+n))\\&\qquad=\sum_{n\in\mathbb{Z}}\frac{u^3-u}{3}\widehat{\xi}(u)\widehat{\eta}(-u).\end{aligned}$$

Here the reality of ξ implies $\overline{\widehat{\xi}(n)}=\widehat{\xi}(-n)$ for its Fourier modes. Moreover we have put $n+m=u, u\in\mathbb{N}$. Then $\sum_{m\geqslant 0}\sum_{n\geqslant 1}\to\sum_{u\geqslant 1}\sum_{n=1}^{u}$ and $\sum_{m+n=u}(u-2n+1)^2=(u^3-u)/3$. Since

$$\begin{aligned}\frac{\mathrm{i}}{3}\int_{-\pi}^{+\pi}\frac{\mathrm{d}\sigma}{2\pi}(\xi'''+\xi')(\sigma)\eta(\sigma)&=\frac{\mathrm{i}}{3}\int_{-\pi}^{+\pi}\frac{\mathrm{d}\sigma}{2\pi}\Big(\sum_{n\in\mathbb{Z}}(-\mathrm{i}u^3+\mathrm{i}u)\mathrm{e}^{\mathrm{i}n\sigma}\widehat{\xi}(u)\Big)\Big(\sum_{q\in\mathbb{Z}}\widehat{\eta}(q)\mathrm{e}^{\mathrm{i}q\sigma}\Big)\\&=\sum_{u,q}\frac{u^3-u}{3}\widehat{\xi}(u)\widehat{\eta}(q)\int_{-\pi}^{+\pi}\frac{\mathrm{d}\sigma}{2\pi}\mathrm{e}^{\mathrm{i}(n+q)\sigma}\\&=\sum_{u}\frac{u^3-u}{3}\widehat{\xi}(u)\widehat{\eta}(-u)\end{aligned}$$

the assertion follows from $S(\xi,\eta)=(\mathrm{i}/8)\,\mathrm{Tr}(F_P[F_P,D_\xi][F_P,D_\eta])$. □

7.4. A closer look on the charges

With the two-cocycle $S(\cdot,\cdot)$, the Lie bracket in $\mathrm{Vect}(S^1)\oplus\mathbb{R}$ can be written as $[(\xi,\alpha),(\eta,\beta)]=([\xi,\eta],S(\xi,\eta))$, where $\eta,\xi\in\mathrm{Vect}(S^1)$ and $\alpha,\beta\in\mathbb{R}$. Then the previously mentioned algebraic cross section coincides with the continuous injection $\xi\mapsto(\xi,0)$. Since $\mathrm{Vect}(S^1)$ is equipped with the Fréchet topology inherited from the underlying space of C^∞ functions, we may ask in what sense $\mathrm{dQ}_P(D_\xi)$ depends on ξ. Since D_ξ is unbounded it will be more appropriate to consider the $\mathbb{R}$-linear map $\xi\mapsto\mathrm{dQ}_P(D_\xi)\varphi$, where $\varphi\in\mathrm{dom}\,D_\xi$.

Proposition 7.3. Let $\varphi=\Psi_P(f_1)\ldots\Psi_P(f_n)\Omega_P$, $f_j\in\mathcal{H}_{\mathrm{smooth}}$ for all $1\leqslant i\leqslant n$. Then the $\mathbb{R}$-linear map $\xi\mapsto\mathrm{dQ}_P(D_\xi)\varphi$ is continuous.

Proof. By $\mathbb{R}$-linearity of $\xi\mapsto\mathrm{dQ}_P(D_\xi)\varphi$ it is sufficient to prove $\|\mathrm{dQ}_P(D_{\xi_n})\varphi\|\to 0$ for any sequence of vector fields (ξ_n) converging to 0. Since $[\mathrm{dQ}_P(D_{\xi_n}),\Psi_P(f)]=\Psi_P(D_{\xi_n}f)$, we have

$$\|\mathrm{dQ}_P(D_{\xi_n})\varphi\|\leqslant\sum_{j=1}^{\infty}M\|f_j\|^{-1}\|D_{\xi_n}f_j\|+M\|\mathrm{dQ}_P(D_{\xi_n})\Omega_P\|$$

where $M=\prod_{j=1}^{n}\|f_j\|$, and where we have used $\|\Psi_P(f)\|\leqslant\|f\|$. From the proof of proposition 4.6 it follows that $\xi_n\to 0$ implies $\|D_{\xi_n}f_j\|\to 0$. Moreover the last term on the right-hand side is estimated with the help of (7.3), and

$$\begin{aligned}\|\mathrm{d}Q_P(D_{\xi_n})\varphi\|^2 &= \tfrac{1}{2}\|PD_\xi(\mathbb{1}-P)\|_{\mathrm{HS}}\\ &\leqslant \tfrac{1}{2}\langle\xi'',\xi''\rangle \qquad \text{by proposition 7.1}\\ &= \frac{1}{2}\int_{-\pi}^{+\pi}\frac{\mathrm{d}\sigma}{2\pi}|\xi_n''(\sigma)|^2 =\leqslant \frac{1}{2}\|\xi_n''\|_\infty .\end{aligned}$$

Thus the last term also vanishes, since $\xi_n \to 0$ then implies $\|\xi_n''\|_\infty \to 0$. □

For simplicity of notation let $\mathrm{d}Q_P(D_\xi) =: \Theta(\xi)$. Then proposition 7.3 implies that $\Theta(\,\cdot\,)\varphi$ is a continuous real linear functional on $\mathrm{Vect}(S^1)$ which has a unique complex linear extension to $\mathrm{Vect}(S^1)_{\mathbb{C}} := \mathrm{Vect}(S^1)\otimes_{\mathbb{R}}\mathbb{C}$. By taking inner products, this functional is then pulled back to a distribution $(\Theta_{\varphi'\varphi}|\,\cdot\,) := \langle\varphi',\Theta(\,\cdot\,)\varphi\rangle$ on $\mathrm{Vect}(S^1)_{\mathbb{C}}$, and we shall denote with $\mathcal{T}$ the subset of all distributions in $\mathrm{Vect}(S^1)^*_{\mathbb{C}}$ of the form $(\Theta_{\varphi'\varphi}|\,\cdot\,)$. If $\xi \in \mathrm{Vect}(S^1)^*_{\mathbb{C}}$, then $(\Theta_{\varphi'\varphi}|\xi)$ can be formally written as an integral with kernel $\Theta_{\varphi'\varphi}(\sigma)$, and it might be useful to have an explicit expression for the integral kernel. Let $\Psi_P(\sigma) := \sum_{n\in\mathbb{Z}}\Psi_P(e_m)\mathrm{e}^{-\mathrm{i}m\sigma}$, then $\Psi_P(f) = \int_{-\pi}^{+\pi}(\mathrm{d}\sigma/2\pi)\Psi_P(\sigma)f(\sigma)$. By (7.3) we find

$$\begin{aligned}(\Theta_{\varphi'\varphi}|\xi) &= \frac{1}{2}\sum_{n,m}\langle\varphi',:\Psi_P(e_n)\langle e_n, D_\xi e_m\rangle\Psi_P(e_m)^*:\varphi\rangle\\ &= \frac{1}{2}\int_{-\pi}^{+\pi}\frac{\mathrm{d}\sigma}{2\pi}\left\langle\varphi',:\Psi_P(\sigma)\left(\mathrm{i}\xi(\sigma)\frac{\mathrm{d}}{\mathrm{d}\sigma}+\frac{\mathrm{i}}{2}\xi'(\sigma)-\frac{\mathrm{i}}{2}\xi(\sigma)\right)\Psi_P(\sigma):\varphi\right\rangle\\ &= \frac{1}{2}\int_{-\pi}^{+\pi}\frac{\mathrm{d}\sigma}{2\pi}\xi(\sigma)\left\langle\varphi',\frac{\mathrm{i}}{2}:\Psi_P(\sigma)\frac{\overleftrightarrow{\mathrm{d}}}{\mathrm{d}\sigma}\Psi_P(\sigma):\varphi\right\rangle\\ &= \frac{1}{2}\int_{-\pi}^{+\pi}\frac{\mathrm{d}\sigma}{2\pi}\xi(\sigma)\Theta_{\varphi'\varphi}(\sigma).\end{aligned} \tag{7.4}$$

Here

$$\Theta_{\varphi'\varphi}(\sigma) := \left\langle\varphi',\frac{\mathrm{i}}{2}:\Psi_P(\sigma)\frac{\overleftrightarrow{\mathrm{d}}}{\mathrm{d}\sigma}\Psi_P(\sigma):\varphi\right\rangle$$

is the formal expression for the integral kernel. It is clear that this kernel is the matrix element of the normal ordered second quantized on-shell expression of the classical energy–momentum tensor density $\Theta_{\pm\pm} = (\mathrm{i}/2)\psi_\pm\overleftrightarrow{\partial}_\pm\psi_\pm$, and $\Theta(\xi)$ is the normal ordered integral over the Noether current $\Theta_{\pm\pm}(\sigma)\xi^\pm(\sigma)$. Therefore we shall refer to the self-adjoint generator $\Theta(\xi)$ as the light cone components of the energy–momentum tensor smeared with the vector field ξ, which obeys the relations

$$\mathrm{i}[\Theta(\xi),\Theta(\eta)] = \Theta([\xi,\eta]) - \frac{1}{24}\left(\int_{-\pi}^{+\pi}\frac{\mathrm{d}\sigma}{2\pi}(\xi'''+\xi')(\sigma)\eta(\sigma)\right)\mathbb{1}. \tag{7.5}$$

8. The anomalous transformation law

8.1. The adjoint action of $Diff_+(S^1)$ on $Vect(S^1) \oplus \mathbb{R}$

The occurrence of a Schwinger term on the right-hand side of (7.5) gives rise to the anomalous transformation law of the energy–momentum tensor in two-dimensional conformal quantum field theory. Here this transformation law is determined by the adjoint action $(Q_P(U_\tau), \Theta(\xi)) \mapsto Q_P(U_\tau)\Theta(\xi)Q_P(U_\tau)^*$ of $(\mathrm{Diff}_+^\sim(S^1))_{\mathrm{ext}}$ on its Lie algebra $\mathrm{Lie}(\mathrm{Diff}_+^\sim(S^1))_{\mathrm{ext}}$. Since the centre of $(\mathrm{Diff}_+^\sim(S^1))_{\mathrm{ext}}$ acts trivially on the latter, and $\mathrm{Lie}(\mathrm{Diff}_+^\sim(S^1))_{\mathrm{ext}}$ is isomorphic to $\mathrm{Vect}(S^1) \oplus \mathbb{R}$, this action factorizes to the adjoint action of $\mathrm{Diff}_+(S^1)$ on $\mathrm{Vect}(S^1) \oplus \mathbb{R}$, which we denote by $\widehat{\mathrm{Ad}}$. For one-parameter subgroups $[Fl^\xi_t]$ in $\mathrm{Diff}_+(S^1)$, $\widehat{\mathrm{Ad}}_{[\mathrm{Fl}^\xi_t]}$ is obtained by direct computation.

Lemma 8.1. Let $\xi, \eta \in \mathrm{Vect}(S^1), \alpha \in \mathbb{R}$. Then $(\eta, \alpha) \in \mathrm{Vect}(S^1) \oplus \mathbb{R}$ and

$$\widehat{\mathrm{Ad}}_{[\mathrm{Fl}^\xi_t]}(\eta, \alpha) = \left(\mathrm{Ad}_{[\mathrm{Fl}^\xi_t]}(\eta), \int_0^t \mathrm{d}s\, S(\xi, \mathrm{Ad}_{[\mathrm{Fl}^\xi_s]}(\eta)) + \alpha\right).$$

Proof. The adjoint action of $\mathrm{Vect}(S^1)$ on its central extension is determined by the Schwinger term, and we have $\widehat{\mathrm{ad}}_\xi(\eta, \alpha) = (\mathrm{ad}_\xi(\eta), S(\xi, \eta))$. Then with the convention $\widehat{\mathrm{ad}}^{(n)} := \widehat{\mathrm{ad}} \circ \cdots \circ \widehat{\mathrm{ad}}$ (n-times) and $\mathrm{ad}_\xi^{(0)}(\eta) := \eta, \widehat{\mathrm{ad}}_\xi^{(0)}(\eta, \alpha) := (\eta, \alpha)$, it follows by induction that $\widehat{\mathrm{ad}}_\xi^{(n)}(\eta) = (\mathrm{ad}_\xi^{(n)}(\eta), S(\xi, \mathrm{ad}_\xi^{(n-1)}(\eta))$. Inserting this identity in the expression for $\widehat{\mathrm{Ad}}$ we obtain

$$\begin{aligned}
\widehat{\mathrm{Ad}}_{[\mathrm{Fl}^\xi_t]}(\eta, \alpha) &= \sum_{n=0}^\infty \frac{t^n}{n!} \widehat{\mathrm{ad}}_\xi^{(n)}(\eta, \alpha) \\
&= \left(\sum_{n=0}^\infty \frac{t^n}{n!} \mathrm{ad}_\xi^{(n)}(\eta), \sum_{n=1}^\infty \frac{t^n}{n!} S(\xi, \mathrm{ad}_\xi^{(n-1)}(\eta)) + \alpha\right) \\
&= \left(\mathrm{Ad}_{[\mathrm{Fl}^\xi_t]}(\eta), \int_0^t \mathrm{d}s\, S(\xi, \sum_{k=0}^\infty \frac{s^k}{k!} \mathrm{ad}_\xi^{(k)}(\eta) + \alpha\right) \\
&= \left(\mathrm{Ad}_{[\mathrm{Fl}^\xi_t]}(\eta), \int_0^t \mathrm{d}s\, S(\xi, \mathrm{Ad}_{[\mathrm{Fl}^\xi_s]}(\eta)) + \alpha\right). \qquad \square
\end{aligned}$$

Now $\int_0^t \mathrm{d}s\, S(\xi, \mathrm{Ad}_{[\mathrm{Fl}^\xi_s]}(\eta))$ can be rewritten in terms of a third-order non-linear differential operator $\Delta : \widetilde{\mathrm{Diff}}_+(S^1) \longrightarrow C^\infty(S^1, \mathbb{R})$ which is closely related to the Schwarzian derivative and whose kernel completely characterizes the universal covering of the $\mathrm{SU}(1,1)/\mathbb{Z}_2$ subgroup in $\mathrm{Diff}_+(S^1)$ (Segal 1987). Let $\tau \in \widetilde{\mathrm{Diff}}_+(S^1)$ then $\Delta : \widetilde{\mathrm{Diff}}_+(S^1) \longrightarrow C^\infty(S^1, \mathbb{R})$ is defined by

$$\Delta(\tau) := \frac{\tau'''}{\tau'} - \frac{3}{2}\left(\frac{\tau''}{\tau'}\right)^2 + \frac{1}{2}((\tau')^2 - 1). \tag{8.1}$$

Note that by definition of Δ we have $\Delta(\tau_1 \circ \tau_2) = (\tau_2')^2 \Delta(\tau_1) \circ \tau_2 + \Delta(\tau_2)$ for any $\tau_1, \tau_2 \in \widetilde{\mathrm{Diff}}_+(S^1)$. Moreover if $[\tau] \in \mathrm{SU}(1,1)/\mathbb{Z}_2$ we observe $\Delta(\tau) = 0$, and with (8.1) we are prepared to prove the following lemma.

Lemma 8.2. Let $\xi,\eta\in\mathrm{Vect}(S^1)$. Then

$$\int_0^1 \mathrm{d}s\, S(\xi,\mathrm{Ad}_{[\mathrm{Fl}^\xi{}_s]}(\eta)) = -\frac{1}{24}\int_{-\pi}^{+\pi}\frac{\mathrm{d}\sigma}{2\pi}\Delta(\mathrm{Fl}^\xi{}_1)(\sigma)\eta(\sigma).$$

Proof. We recall that if $\tau\in\widetilde{\mathrm{Diff}}_+(S^1)$ and $\eta\in\mathrm{Vect}(S^1)$ then $\mathrm{Ad}_{[\tau]}(\eta)=[\tau]_*(\eta)$ where $[\tau]_*$ is the push forward map for vector fields: $([\tau]_*(\eta))(\sigma)(\mathrm{d}/\mathrm{d}\sigma)=\tau'\circ\tau^{-1}(\sigma)\eta\circ\tau^{-1}(\sigma)(\mathrm{d}/\mathrm{d}\sigma)$. For simplicity of notation we put $\mathrm{Fl}^\xi{}_s =: \tau_s$. Then

$$\int_0^1 \mathrm{d}s\, S(\xi,\mathrm{Ad}_{[\mathrm{Fl}^\xi{}_s]}(\eta))$$
$$= -\frac{1}{24}\int_{-\pi}^{+\pi}\frac{\mathrm{d}\sigma}{2\pi}\int_0^1 \mathrm{d}s\,\tau_s'\circ\tau_{-s}(\sigma)\eta\circ\tau_{-s}(\sigma)\left\{\frac{\mathrm{d}^3}{\mathrm{d}\sigma^3}+\frac{\mathrm{d}}{\mathrm{d}\sigma}\right\}\xi(\sigma).$$

With the substitution $\tau_{-s}(\sigma)=\vartheta$ we obtain

$$\mathrm{d}\vartheta=\tau_s'(\sigma)\mathrm{d}\sigma,\ \frac{\mathrm{d}}{\mathrm{d}\sigma}=\tau_{-s}'\circ\tau_s(\vartheta)\frac{\mathrm{d}}{\mathrm{d}\vartheta}=:\mathcal{D}_s$$

where $\mathcal{D}_s$ is a differential operator in ϑ. By definition we have $\tau_s=\mathrm{Fl}^\xi{}_s$ such that $\xi\circ\tau_s(\vartheta)=\dot\tau_s(\vartheta)$ and the right-hand side in this display reads

$$-\frac{1}{24}\int_{\tau_{-s}(-\pi)}^{\tau_{-s}(+\pi)}\frac{\mathrm{d}\vartheta}{2\pi}\eta(\vartheta)\int_0^1\mathrm{d}s\,(\tau_s'(\vartheta))^2(\mathcal{D}_s\circ\mathcal{D}_s\circ\mathcal{D}_s\dot\tau_s+\mathcal{D}_s\dot\tau_s)(\vartheta).$$

Using the chain rule, $\mathcal{D}_s\circ\mathcal{D}_s\circ\mathcal{D}_s\dot\tau_s$ and $\mathcal{D}_s\dot\tau_s$ can be easily computed. In terms of the derivatives of τ_s and τ_{-s} the s-integrand reads

$$(\tau_s')^2(\mathcal{D}_s\circ\mathcal{D}_s\circ\mathcal{D}_s\dot\tau_s+\mathcal{D}_s\dot\tau_s)$$
$$=(\tau_{-s}''\circ\tau_s)^2(\tau_s')^3\dot\tau_s{}'+\tau_{-s}'''\circ\tau_s\dot\tau_s{}'(\tau_s')^2+\tau_{-s}''\circ\tau_s\tau_s''\dot\tau_s{}'$$
$$+2\tau_{-s}''\circ\tau_s\tau_s'\dot\tau_s{}''+(1/\tau_s')\dot\tau_s{}'''+\tau_{-s}''\circ\tau_s\tau_s'\dot\tau_s{}''$$
$$=\frac{\dot\tau_s{}'''}{\tau_s'}-\frac{\tau_s'''}{(\tau_s')^2}+3\frac{\tau_s''\dot\tau_s{}'}{(\tau_s')^3}-2\frac{(\tau_s''\dot\tau_s)''}{(\tau_s)^2}-\frac{\tau_s''\dot\tau_s{}''}{(\tau_s')^2}+\tau_s'\dot\tau_s{}'=:(\star).$$

Here we have replaced the derivatives $\tau_{-s}'\circ\tau_s,\tau_{-s}''\circ\tau_s$ and $\tau_{-s}'''\circ\tau_s$ by $1/\tau_s, -\tau_s''/(\tau_s')^3$ and $(3(\tau_s'')^2-(\tau_s''')\tau_s')/(\tau_s')^5$. This follows by differentiating the identity $\tau_{-s}\circ\tau_s(\sigma)=\sigma$ with respect to σ. Finally $(\star)$ can be rewritten as a total derivative with respect to s:

$$\int_0^1\mathrm{d}s\,S(\xi,\mathrm{Ad}_{[\mathrm{Fl}^\xi{}_t]}(\eta))$$
$$=-\frac{1}{24}\int_{-\pi}^{+\pi}\frac{\mathrm{d}\sigma}{2\pi}\int_0^1(\star)(\sigma)\eta(\sigma)$$
$$=-\frac{1}{24}\int_{-\pi}^{+\pi}\frac{\mathrm{d}\sigma}{2\pi}\int_0^1\mathrm{d}s\frac{\mathrm{d}}{\mathrm{d}s}\left[\frac{\tau_s'''}{\tau_s'}-\frac{3}{2}\left(\frac{\tau_s''}{\tau_s'}\right)^2+\frac{1}{2}((\tau_s')^2-1)\right](\sigma)\eta(\sigma)$$
$$=-\frac{1}{24}\int_{-\pi}^{+\pi}\frac{\mathrm{d}\sigma}{2\pi}\left(\frac{\tau_s'''}{\tau_s'}-\frac{3}{2}\left(\frac{\tau_s''}{\tau_s'}\right)^2+\frac{1}{2}((\tau_s)^2-1)\right)(\sigma)\eta(\sigma)$$
$$=-\frac{1}{24}\int_{-\pi}^{+\pi}\frac{\mathrm{d}\sigma}{2\pi}\Delta(\mathrm{Fl}^\xi{}_1)(\sigma)\eta(\sigma).$$

□

By lemmas 8.1 and 8.2, the adjoint action of $\mathrm{Diff}_+(S^1)$ on $\mathrm{Vect}(S^1)\oplus\mathbb{R}$ is given by

$$\widehat{\mathrm{Ad}}_{[\tau]}(\eta,\alpha) = ([\tau]_*\eta, \alpha - \frac{1}{24}\int_{-\pi}^{+\pi}\frac{\mathrm{d}\sigma}{2\pi}\Delta(\tau)(\sigma)\eta(\sigma))$$

and, taking the discussion preceding lemma 8.1 into account, the anomalous transformation law of the smeared energy–momentum tensor components under the adjoint action of $(\mathrm{Diff}_+^\sim(S^1))_{\mathrm{ext}}$ reads

$$Q_P(U_\tau)\Theta(\eta)Q_P(U_\tau)^* = \Theta([\tau]_*\eta) - \frac{1}{24}\left(\int_{-\pi}^{+\pi}\frac{\mathrm{d}\sigma}{2\pi}\Delta(\tau)(\sigma)\eta(\sigma)\right)\mathbb{1}. \tag{8.2}$$

We recall that this action factorizes to an action of $\mathrm{Diff}_+(S^1)$ on $\mathrm{Lie}(\mathrm{Diff}_+^\sim(S^1))_{\mathrm{ext}}$.

8.2. Connections to geometry

On the level of classical field theory, it is known that the light cone components $\Theta_{\pm\pm}[\psi_\pm]$ of the energy–momentum tensor of the theory can be considered from a more geometrical point of view as elements in an affine hyperplane in the space of extended quadratic differentials on the circle (Segal 1981). Then these components transform according to the restriction of the inverse of the coadjoint action of $\mathrm{Diff}_+(S^1)$ on this hyperplane. In the following we shall show that these structures arise in a natural way within our algebraic treatment. To do so we must introduce some notation.

Let $\mathcal{T}_2^0(S^1)$ be the bundle of contravariant tensors of rank 2 over S^1 and let $\mathcal{Q}$ be the vector space of smooth sections $S^1 \to \mathcal{T}_2^0(S^1)$. We shall refer to $\mathcal{Q}$ as the space of quadratic differentials. Since $\mathrm{d}\sigma\otimes\mathrm{d}\sigma$ is a global frame in $\mathcal{T}_2^0(S^1)$, $\mathcal{Q}$ can be identified with $C^\infty(S^1,\mathbb{R})$. Moreover any $\xi \in \mathrm{Vect}(S^1)$ induces via insertion a complex linear map $\iota_\xi : \mathcal{Q} \to \Omega^1(S^1)$, where ι_ξ is given by $(\iota_\xi \circ X)(\sigma)\,\mathrm{d}\sigma := \xi(\sigma)X(\sigma)\,\mathrm{d}\sigma$, $X \in \mathcal{Q}$ and $\Omega^1(S^1)$ is the space of 1-forms over S^1. Integration of $\iota_\xi \circ X$ over S^1 yields a real number, which defines a bilinear map $(\cdot|\cdot) : \mathcal{Q} \times \mathrm{Vect}(S^1) \to \mathbb{R}$

$$(X,\xi) \longmapsto (X|\xi) =: \int_{S^1}\iota_\xi \circ X. \tag{8.3}$$

This map extends canonically to a bilinear map $\mathcal{Q} \times \mathrm{Vect}(S^1)_{\mathbb{C}} \to \mathbb{C}$, which is complex linear in the second entry by $(X|\xi + \mathrm{i}\eta) := (X|\xi) + \mathrm{i}(X|\eta)$, where ξ,η are real. Since by definition $(X|\cdot)$ is a bounded linear functional on $\mathrm{Vect}(S^1)_{\mathbb{C}}$, such a quadratic differential determines uniquely a distribution in $\mathrm{Vect}(S^1)^*_{\mathbb{C}}$. Hence there exists a linear injection $j : \mathcal{Q} \to \mathrm{Vect}(S^1)^*_{\mathbb{C}}$, $j(X) := (X|\cdot)$, which to identify via (8.3) the elements $X \in \mathcal{Q}$ to be identified with distributions $j(X)$.

The connection of our algebraic treatment with the geometic formulation is established by restricting attention to a special subset $\mathcal{T}^{\mathbb{R}}_{\mathrm{smooth}}$ of distributions in $\mathcal{T}$, which are real-valued when evaluated on real vector fields, and which correspond uniquely to C^∞ functions on S^1. To be more precise let $\mathcal{T}^{\mathbb{R}}_{\mathrm{smooth}}$ be the subset of distributions $(\Theta_{\varphi'\varphi}|\cdot)$ in $\mathcal{T}$ for which $\varphi' = \varphi$, $\|\varphi\| = 1$, and which have smooth kernel

$$\frac{\mathrm{i}}{2}\left\langle \varphi, :\Psi_P(\sigma)\frac{\overleftrightarrow{\mathrm{d}}}{\mathrm{d}\sigma}\Psi_P(\sigma):\varphi\right\rangle.$$

Then $\mathcal{T}^{\mathbb{R}}_{\text{smooth}} \neq \emptyset$, since for $\varphi = \Psi_P(e_{n_1}) \ldots \Psi_P(e_{n_k})\Omega_P, n_j \geqslant 1, 1 \leqslant j \leqslant k$, the kernel $\Theta_{\varphi\varphi}(\sigma)$ is a real-valued trigonometric polynomial and hence analytic. Moreover we observe that, by definition $\mathcal{T}^{\mathbb{R}}_{\text{smooth}} \subset j(\mathcal{Q})$ and by injectivity of j, these $(\Theta_{\varphi\varphi}|\cdot)$ uniquely correspond to quadratic differentials on S^1 which we shall denote by $\Theta_{\varphi\varphi}$. $\Theta_{\varphi\varphi}$ like these are then given by

$$\frac{1}{2\pi}\left\langle \varphi, : \Psi_P(\sigma) \frac{\overleftrightarrow{\mathrm{d}}}{\mathrm{d}\sigma} \Psi_P(\sigma) : \varphi \right\rangle \mathrm{d}\sigma \otimes \mathrm{d}\sigma.$$

The $(\mathrm{Diff}^{\sim}_{+}(S^1))_{\text{ext}}$ action (8.2) on $\mathrm{Lie}(\mathrm{Diff}^{\sim}_{+}(S^1))_{\text{ext}}$, which factorizes to the adjoint action of $\mathrm{Diff}_+(S^1)$ on $\mathrm{Vect}(S^1) \oplus \mathbb{R}$ automatically induces a right action $\mathcal{R} : \mathcal{T} \times \mathrm{Diff}_+(S^1) \to \mathcal{T}$ on the family of distributions $\mathcal{T}$, which is obtained by inserting (8.2) into $\langle \varphi', \cdot\, \varphi \rangle$. In fact for any $[\tau]$ in $\mathrm{Diff}_+(S^1)$ we have

$$\mathcal{R}_{[\tau]}\big((\Theta_{\varphi\varphi}|\cdot)\big) = (\Theta_{Q_P(U_\tau)^*\varphi Q_P(U_\tau)\varphi}|\cdot) \tag{8.4}$$

and $\mathcal{R}_{[\tau_1]} \circ \mathcal{R}_{[\tau_2]} = \mathcal{R}_{[\tau_1 \circ \tau_2]}$ is easily verified. Moreover $\mathcal{R}$ leaves $\mathcal{T}^{\mathbb{R}}_{\text{smooth}}$ invariant, since a $(\Theta_{\varphi\varphi}|\cdot)$ in $\mathcal{T}^{\mathbb{R}}_{\text{smooth}}$ has $\|\varphi\| = 1$ with a smooth kernel $\Theta_{\varphi\varphi}(\sigma)$. But from the left-hand side of (8.2) we conclude that $(\Theta_{Q_P(U_\tau)^*\varphi Q_P(U_\tau)^*\varphi}|\cdot)$ has a smooth kernel $\tau'(\sigma)^2\Theta_{\varphi\varphi} \circ \tau(\sigma) - \frac{1}{24}\Delta(\tau)(\sigma)$ which together with $\|Q_P(U_\tau)^*\varphi\| = 1$ implies $\mathcal{R}$ invariance of $\mathcal{T}^{\mathbb{R}}_{\text{smooth}}$. This right action has an illuminating geometrical interpretation.

Proposition 8.3. Let $[\tau] \in \mathrm{Diff}_+(S^1)$, let $X \in \mathcal{Q}$, and let $R_{[\tau]} : \mathcal{Q} \to \mathcal{Q}$ be the affine map $X \mapsto R_{[\tau]}X := [\tau]^*X - \frac{1}{12}\Delta(\tau)$, where $[\tau]^*$ is the pull back of contravariant tensors under the diffeomorphism $[\tau]$ and where $\Delta(\tau)$ denotes the quadratic differential $\frac{1}{2}\pi\Delta(\tau)\mathrm{d}\sigma \otimes \mathrm{d}\sigma$ in $\mathcal{Q}$ which is associated with each $\tau \in \widetilde{\mathrm{Diff}}_+(S^1)$. Then

$$R : \mathcal{Q} \times \mathrm{Diff}_+(S^1) \longrightarrow \mathcal{Q}$$

is an affine right action of $\mathrm{Diff}_+(S^1)$ on $\mathcal{Q}$. Moreover let $\widetilde{\mathcal{Q}}$ denote the pre-image of $\mathcal{T}^{\mathbb{R}}_{\text{smooth}}$ under the injection $j : \mathcal{Q} \to \mathrm{Vect}(S^1)^*_{\mathbb{C}}$, then the following diagram commutes:

$$\begin{array}{ccc} \widetilde{\mathcal{Q}} & \xrightarrow{R_{[\tau]}} & \widetilde{\mathcal{Q}} \\ j\downarrow & & \downarrow j \\ \mathcal{T}^{\mathbb{R}}_{\text{smooth}} & \xrightarrow{\mathcal{R}_{[\tau]}} & \mathcal{T}^{\mathbb{R}}_{\text{smooth}}. \end{array}$$

Proof. By (7.8) we have for $\tau_1, \tau_2 \in \widetilde{\mathrm{Diff}}_+(S^1)$ the identity $\Delta(\tau_1 \circ \tau_2) = (\tau_2')^2\Delta(\tau_1) \circ \tau_2 + \Delta(\tau_2)$, which together with $[\tau_1 \circ \tau_2]^* = [\tau_2]^* \circ [\tau_1]^*$ implies $R_{[\tau_1 \circ \tau_2]} = R_{[\tau_2]} \circ R_{[\tau_1]}$. Thus R is an affine right action. Moreover let $\Theta_{\varphi\varphi} \in \widetilde{\mathcal{Q}}$, then $R_{[\tau]}\Theta_{\varphi\varphi}$ can be written as $\frac{1}{2}\pi((\tau')^2\Theta_{\varphi\varphi} \circ \tau - \frac{1}{24}\Delta(\tau))(\sigma)\mathrm{d}\sigma \otimes \mathrm{d}\sigma$ by definition of $[\tau]^*$, and the expression inside the bracket is exactly the integral kernel of $(\Theta_{Q_P(U_\tau)^*\varphi Q_P(U_\tau)^*\varphi}|\cdot)$. This proves proposition 8.3. $\square$

For the sake of completeness it should be mentioned that—following Segal (1981)—the affine right action of $\mathrm{Diff}_+(S^1)$ on Q is obtained as follows. Let $Q \oplus \mathbb{R}$ be the real vector space of extended quadratic differentials containing the hyperplane $Q \oplus 1 := \{(X, 1) : X \in Q\}$ as a closed subspace. Then Q is canonically identified by $X \mapsto (X, 1)$ with $Q \oplus \mathbb{R}$. Moreover each (X, α) acts on $\mathrm{Vect}(S^1) \oplus \mathbb{R}$ as a bounded real linear functional $((X, \alpha)| \cdot)$, where $((X, \alpha)|(\xi, \beta)) := (X|\xi) + \alpha\beta$ for all (ξ, β) in $\mathrm{Vect}(S^1) \oplus \mathbb{R}$. Hence $Q \oplus \mathbb{R}$ lies in the dual of $\mathrm{Vect}(S^1) \oplus \mathbb{R}$, and the adjoint action $\widehat{\mathrm{Ad}}$ of $\mathrm{Diff}_+(S^1)$ on $\mathrm{Vect}(S^1) \oplus \mathbb{R}$ induces by transposition a right action $\widehat{\mathrm{Ad}}^t$ on the dual, which is exactly the inverse of the coadjoint action. From the definition of $\widehat{\mathrm{Ad}}$ it follows that $(\widehat{\mathrm{Ad}}_{[\tau]}{}^t(X, 1)|(\xi, \beta) = ((X, 1)|\widehat{\mathrm{Ad}}_{[\tau]}(\xi, \beta)) = ((R_{[\tau]}X, 1)|(\xi, \beta))$, which then shows that $\widehat{\mathrm{Ad}}_{[\tau]}{}^t(X, 1) = (R_{[\tau]}X, 1)$.

Note added in proof. I owe special thanks to Professor B Schroer for raising my attention to earlier work on the construction of the order–disorder variables with operator methods (Schroer and Truong 1978). The global operator expansions in conformally invariant QFT as treated in Schroer *et al* (1978), and the conformal blocks were not an invention of the 1980s. Their existence and decomposition theory has been well known since 1974/75 (resolving the causality paradox in two-dimensional conformal QFT by operator methods). However no non-trivial model (non-Abelian statistics) was known until the Coulomb representations of Kadanoff, Nienkius, De Nies and BPZ. It would be interesting to recover in the previous approach the explicit n-point functions of the chiral $d = 1/16$ components which have been explicitly computed in Rehren and Schroer (1988) by holomorphic factorization of the doubled model. Moreover I am indebted to Professor H Grosse and to Professor K Fredenhagen for encouragement and help.

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